CORNEL BĂEȚICA

DETERMINANTAL IDEALS

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Referenți științifici: Prof. dr. **C. Niță**
Conf. dr. **Tiberiu Dumitrescu**

Șos. Panduri, 90-92, București - 76235; Telefon/Fax: 410.23.84
E-mail: editura@unibuc.ro
Internet: www.editura.unibuc.ro

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Introduction

This book grew up from the lectures we held to the graduate students at the Bucharest University in the last years. Of course, the book goes beyond the lectures themselves, but we tried here to be as complete as possible. Implicitly we suppose that the reader is familiar with the basics of commutative algebra and though we include some of the main notions and results from Bruns and Vetter' book [21], a reference in the area, we recommend the interested reader to take a look to this book. Actually, only the first two chapters of our book follow the line of investigation of Bruns and Vetter [21], the rest relying on the method started by Sturmfels [61].

The study of determinantal ideals, respectively of classical determinantal rings is an old topic of commutative algebra. As in most of the cases the theory evolved from algebraic geometry, but became soon an important topic of commutative algebra. Nowadays the field is still under investigation and there are many yet unsolved problems.

Let us give a brief description of our book.

Chapter 1 is mainly expository, with a few proofs, but here enters the scene the most important notions that will be used throughout the book. We start with the definition of graded algebras with straightening law on a poset (doset) over an arbitrary commutative ring. Then we split the exposition into three parts that correspond to the three types of matrices we are interested in. In part (G) we study generic matrices, in part (S) generic symmetric matrices, and in part (A) generic alternating matrices. This kind of presentation will be encountered throughout the book.

Chapter 2 deals with the computation of the divisor class group and canonical class of determinantal rings which correspond to 1-cogenerated ideals. Finally we determine the Gorenstein rings among the rings under consideration.

Chapter 3 is the core of the entire book. We start by describing the combinatorial algorithms INSERT and DELETE, and then the Knuth-Robinson-Schensted correspondence, KRS for short, between standard (Young) bitableaux and two-line arrays of positive integers of a certain type. The theorems of Schensted and Greene are reviewed in the next section. The theorem of Schensted [58] deals with the determination of the length of the longest increasing (decreasing) subsequence of a given sequence of integers. In fact, the length of the longest increasing (resp. decreasing) subsequence of a given sequence v is the length of the first row (resp. column) in the tableau INSERT(v). An interpretation of the rest of the shape of INSERT(v) is given by Greene's theorem [43]. Next we describe a KRS type correspondence between standard monomials and ordinary monomials of a set of indeterminates, and at the end we use this in order to determine Gröbner bases for all three types of determinantal ideals.

In Chapter 4 we first recall some results on the primary decomposition of the powers (products) of determinantal ideals. It turns out that the primary decomposition depends on the characteristic of the field we are working over. Next we determine Gröbner bases for the powers of ideals of maximal minors (pfaffians) without reference to the characteristic. In the last section we exploit the theorems of Schensted and Greene in order to get Gröbner bases for the (symbolic) powers of determinantal ideals.

Chapter 5 opens a new perspective in the study of determinantal ideals based on the principle of deriving properties of ideals and algebras from their initial counterparts. In fact, we associate to the initial ideal of any determinantal ideal a shellable simplicial complex such that the corresponding residue class ring of the initial ideal is the Stanley-Reisner ring of the complex. Then some of the combinatorial properties of simplicial complexes are interpreted in terms of families of lattice paths of a certain type, and thus the determination of the multiplicity, of the Hilbert series or of the a-invariant of determinantal rings become a counting paths matter.

All these techniques are brought together in Chapter 6 to contribute to the investigation of Rees algebras of determinantal (pfaffian) ideals and of algebras of minors (pfaffians). In the first section we show that the Rees algebras of determinantal (pfaffian) ideals and the algebras of minors (pfaffians) are Cohen-Macaulay and normal domains in non-exceptional characteristic. The next sections are devoted to the description of the divisor class group and the canonical class of Rees algebras of determinantal (pfaffian) ideals and of algebras of minors (pfaffians). Finally we also determine the Gorenstein rings among the rings under consideration.

At the end we would like to thank all the people who, directly or indirectly, helped us to write this book.

Bucharest, May 2003

Cornel Baetica

Chapter 1

Preliminaries

The straightening law of Doubilet, Rota and Stein [33], in the approach of De Concini, Eisenbud and Procesi [29], is an indispensable tool in the study of determinantal rings. From the algebraic point of view, the determinantal rings are graded algebras with straightening law, and this shall provide us some of their main properties.

Definition 1.0.1 Let A be a B-algebra and $\Delta \subseteq A$ a finite subset with a partial order $\preceq$, called a *poset*. Then A is a *graded algebra with straightening law* (for short ASL) on Δ over B if the following conditions hold:

(H_0) $A = \bigoplus_{i\geq 0} A_i$ is a positively graded B-algebra such that $A_0 = B$, Δ consists of homogeneous elements of positive degree and generates A as a B-algebra.

(H_1) The products $\delta_1 \cdots \delta_u$, $u \in \mathbb{N}$, $\delta_i \in \Delta$, such that $\delta_1 \preceq \cdots \preceq \delta_u$ are linearly independent over B. We call them *standard monomials.*

(H_2) (*Straightening law*) For all incomparable $\delta_1, \delta_2 \in \Delta$ the product $\delta_1\delta_2$ has a representation

$$\delta_1\delta_2 = \sum \lambda_\mu \mu, \quad \lambda_\mu \in B, \quad \lambda_\mu \neq 0, \quad \mu \text{ standard monomial },$$

that satisfies the following condition: every standard monomial μ contains a factor $\zeta \in \Delta$ such that $\zeta \preceq \delta_1$, $\zeta \preceq \delta_2$ (we allow that $\delta_1\delta_2 = 0$, then the sum $\sum \lambda_\mu \mu$ being empty).

Actually the standard monomials form a B-basis of A, the *standard basis* of A. The representation of an element of A as a linear combination of standard monomials is called its *standard representation.* The relations in (H_2) will be referred to as the *straightening relations.*

The most interesting examples of ASLs are rings related to matrices and determinants; see De Concini, Eisenbud and Procesi [30], Bruns and Vetter [21].

In order to apply the ASL theory to determinantal rings one has to know that these rings are indeed ASLs. This follows readily from the fact that their defining ideals have a distinguished system of generators with respect to the underlying poset. To be precise, let us consider A an ASL on a poset Δ (over B), and $\Delta' \subseteq \Delta$ a *poset ideal* of Δ, i.e. Δ' contains all elements $x \preceq y$ if $y \in \Delta'$. Set $I = \Delta' A$ the ideal of A generated by Δ'.

Proposition 1.0.2 *The residue class ring A/I is a graded* ASL *on $\Delta \setminus \Delta'$ over B.*

When Δ' is the poset ideal *cogenerated* by a subset Ω of Δ, that is $\Delta' = \{\xi \in \Delta : \omega \not\preceq \xi \text{ for each } \omega \in \Omega\}$, then the ideal $I = \Delta' A$ is called the *ideal cogenerated* by Ω. In particular, when $|\Omega| = 1$ the ideal cogenerated by Ω is said to be a 1-*cogenerated* ideal.

An useful generalization of the notion of graded algebra with straightening law on a poset is that of graded algebra of straightening law on a doset. We define the underlying notions.

Definition 1.0.3 Let H be a finite poset with order $\preceq$. A subset D of $H \times H$ is a *doset* if D satisfies the following conditions:

(a) $(\alpha, \alpha) \in D$ for all $\alpha \in H$;

(b) if $(\alpha, \beta) \in D$, then $\alpha \preceq \beta$;

(c) if $\alpha \preceq \beta \preceq \gamma \in H$, then

$$(\alpha, \gamma) \in D \iff (\alpha, \beta) \in D \text{ and } (\beta, \gamma) \in D.$$

Definition 1.0.4 Let B be a ring, and let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded B-algebra such that $A_0 = B$. Let D be a doset of a poset H, and suppose that $D \subseteq A$. Then A is a *graded algebra with straightening law* on doset D over B (for short DASL) if the following conditions are satisfied:

(H_0) D consists of homogeneous elements of positive degree in A.

(H_1) The products of the form $(\alpha_1, \alpha_2) \cdots (\alpha_{2k-1}, \alpha_{2k})$, $k \geq 1$, $(\alpha_{2i-1}, \alpha_{2i}) \in D$, such that $\alpha_1 \preceq \alpha_2 \preceq \cdots \preceq \alpha_{2k-1} \preceq \alpha_{2k}$ form a B-basis of A. We call them *standard monomials.*

(H_2) For $(\beta_{2i-1},\beta_{2i}) \in D$, $i=1,\dots,l$, let $M=(\beta_1,\beta_2)\cdots(\beta_{2l-1},\beta_{2l})$. Moreover, let

$$M=\sum \lambda_N N,\ \ \lambda_N\in B,\ \ \lambda_N\neq 0,\ \ N \text{ standard monomial },$$

be the representation of M as a linear combination of standard monomials (standard representation). Let $N=(\gamma_1,\gamma_2)\cdots(\gamma_{2h-1},\gamma_{2h})$ be any of the standard monomials appearing on the right side of the equation. Then for all permutations σ of the set $\{1,\dots,2l\}$ one has that the sequence $(\beta_{\sigma(1)},\dots,\beta_{\sigma(2l)})$ is lexicographically greater or equal than the sequence $(\gamma_1,\dots,\gamma_{2h})$.

(H_3) In the notation of (H_2), suppose that there is a permutation σ such that $\beta_{\sigma(1)}\preceq\cdots\preceq\beta_{\sigma(2l)}$. The standard monomial

$$(\beta_{\sigma(1)},\beta_{\sigma(2)})\cdots(\beta_{\sigma(2l-1)},\beta_{\sigma(2l)})$$

must appear with coefficient ± 1 in the standard representation of M.

(**G**) Let $X=(X_{ij})$ be an $m\times n$ matrix of indeterminates over a commutative ring B, for short a *generic matrix*, $\Delta(X)$ the set of all minors of X, and $B[X]=B[X_{ij}:1\le i,j\le n]$ the polynomial ring over B. Consider the set $\Delta(X)$ ordered in the following way:

$$[a_1,\dots,a_t|b_1,\dots,b_t]\preceq[c_1,\dots,c_s|d_1,\dots,d_s] \text{ if and only if } t\ge s \text{ and } a_i\le c_i,\ b_i\le d_i \text{ for } i=1,\dots,s,$$

and let $\delta\in\Delta(X)$, $\delta=[a_1,\dots,a_t|b_1,\dots,b_t]$. As usual, one denotes by $[a_1,\dots,a_t|b_1,\dots,b_t]$ the minor $\det(X_{a_ib_j})_{1\le i,j\le t}$. We define $I(X,\delta)$ to be the ideal generated by all minors which are not greater or equal than δ. Thus the ideal $I(X,\delta)$ is the ideal of $B[X]$ cogenerated by δ. Denote by $R(X,\delta)$ the residue class ring of $B[X]$ with respect to the ideal $I(X,\delta)$ and by $\Delta(X,\delta)$ the set of all minors which are greater or equal than δ.

In particular, if $\delta=[1,\dots,t|1,\dots,t]$ then $I(X,\delta)$ is the ideal $I_{t+1}(X)$ generated by all the $(t+1)$-minors of X, and denote by $R_{t+1}(X)$ the analogous residue class ring. When B is a field we call $R_{t+1}(X)$ the *classical determinantal ring.*

Our investigation of the rings $R(X,\delta)$ is based on the knowledge of their combinatorial structure, see De Concini, Eisenbud and Procesi [30], [29], and on the methods developed by Bruns and Vetter in the fundamental book [21]. In this approach one considers all the minors of X as generators of the B-algebra $B[X]$, and not only 1-minors X_{ij}. So we may interpret the products of minors as "monomials", but the price to be paid is that the computations may become tedious. Therefore one has to choose a proper subset of all these "monomials" as a B-basis: we will see that the *standard* monomials form a B-basis, and the straightening law will help us to express an arbitrary product of minors as a linear combination of standard monomials.

Let $\mu = \delta_1 \cdots \delta_u$ be a monomial, i.e. a product of minors. We can always assume that the sizes of minors δ_i, denoted by $|\delta_i|$, are in non-increasing order $|\delta_1| \geq \cdots \geq |\delta_u|$. By convention, the value of the empty minor $[\,|\,]$ is 1. The *shape* $|\mu|$ of μ is the sequence $(|\delta_1|, \ldots, |\delta_u|)$. Moreover, the monomial μ is said to be *standard* if and only if $\delta_1 \preceq \cdots \preceq \delta_u$.

The combinatorial structure of $B[X]$ with respect to products of minors is clarified by the following theorem; see De Concini, Eisenbud and Procesi [30, pg.51], [29] or Bruns and Vetter [21, Chapter 4].

Theorem 1.0.5 *The ring $B[X]$ is a graded algebra with straightening law on $\Delta(X)$ over B, that is:*
(a) *The standard monomials form a B-basis of $B[X]$.*
(b) *The product of two minors $\delta_1, \delta_2 \in \Delta(X)$ such that $\delta_1\delta_2$ is not a standard monomial has a representation*

$$\delta_1\delta_2 = \sum \lambda_i \xi_i \eta_i, \quad \lambda_i \in B, \quad \lambda_i \neq 0,$$

where $\xi_i\eta_i$ is a standard monomial and $\xi_i \prec \delta_1$, $\delta_2 \prec \eta_i$ (we allow here that η_i is the empty minor).
(c) *The standard representation of an arbitrary monomial μ can be found by successive applications of the straightening relations in* (b).

Some remarks are in order here:

Remark 1.0.6 (a) Usually the proof of Theorem 1.0.5 goes by passing to the subalgebra $B[\Gamma(X)]$ generated by the set $\Gamma(X)$ of maximal minors of X. The advantage of considering $B[\Gamma(X)]$ instead of $B[X]$ stands on the fact that the maximal minors satisfy the famous Plücker relations; see, for instance, Bruns and Vetter [21, Lemma (4.4)]. Notice that the Plücker relations are homogeneous of degree 2 (the maximal minors are considered in $B[\Gamma(X)]$ as having degree 1). Therefore there are exactly two factors ξ_i and η_i in each term on the right side of the equation

$$\delta_1\delta_2 = \sum \lambda_i \xi_i \eta_i,$$

with δ_1, δ_2 maximal minors. But the straightening law in $B[X]$ is a specialization of that in $B[\Gamma(X')]$, where X' is a generic matrix that contains X as a submatrix. This yields a justification for part (b) of the Theorem 1.0.5.

(b) When $m \leq n$ and B is a field, the algebra $B[\Gamma(X)]$ is the homogeneous coordinate ring of the Grassmann variety of the m-dimensional vector subspaces of B^n.

As an immediate consequence of Theorem 1.0.5 and Proposition 1.0.2 we get:

Corollary 1.0.7 *The ring $R(X,\delta)$ is a graded algebra with straightening law on $\Delta(X,\delta)$ over B.*

The sets $\Delta(X)$ and $\Delta(X,\delta)$ are distributive lattices. Now by Bruns and Vetter [21, Theorem (5.14)] we can immediately conclude:

Proposition 1.0.8 *If B is a Cohen-Macaulay ring, then $R(X,\delta)$ is Cohen-Macaulay.*

The following proposition shows how to compute the dimension of the ring $R(X,\delta)$, where $\delta = [a_1,\ldots,a_t|b_1,\ldots,b_t] \in \Delta(X)$.

Proposition 1.0.9 *We have* $\dim R(X,\delta) = \dim B + (m+n+1)t - \sum_{i=1}^{t}(a_i + b_i)$. *In particular, the classical determinantal ring $R_{t+1}(X)$ is Cohen-Macaulay of dimension $(n+m-t)t$.*

(S) Let $X = (X_{ij})$ be an $n \times n$ symmetric matrix of indeterminates over a commutative ring B i.e. $X_{ij} = X_{ji}$ for all $i > j$, for short a *generic symmetric matrix*, and $B[X] = B[X_{ij} : 1 \le i \le j \le n]$ the polynomial ring over B. Let H be the set of the non-empty subsets of $\{1,\ldots,n\}$ and $\alpha = \{\alpha_1,\ldots,\alpha_t\} \in H$ with $\alpha_1 < \cdots < \alpha_t$. One defines $I(X,\alpha)$ as being the ideal of $B[X]$ generated by all the i-minors of the first $\alpha_i - 1$ rows of X, for $i = 1,\ldots,t$, and by all $(t+1)$-minors of X. Denote by $R(X,\alpha)$ the residue class ring of $B[X]$ with respect to the ideal $I(X,\alpha)$.

In particular, if $\alpha = \{1,\ldots,t\}$ then $I(X,\alpha)$ is the ideal $I_{t+1}(X)$ generated by all the $(t+1)$-minors of X, and denote by $S_{t+1}(X)$ the analogous residue class ring. When B is a field we call $S_{t+1}(X)$ the *classical ring of symmetric minors.*

There is also a standard monomial approach to the structure of $R(X,\alpha)$, in which DASLs replace ASLs. On the set H we define the following partial order:

$$a = \{a_1,\ldots,a_t\} \preceq b = \{b_1,\ldots,b_s\} \Longleftrightarrow t \ge s \text{ and } a_i \le b_i \text{ for } i = 1,\ldots,s.$$

Since X is a symmetric matrix, it is obvious that we have $[a_1,\ldots,a_t|b_1,\ldots,b_t] = [b_1,\ldots,b_t|a_1,\ldots,a_t]$. Let $M = M_1 \cdots M_u$ be a monomial, i.e. a product of minors. We can assume that the sizes of minors M_i, denoted by $|M_i|$, are in non-increasing order $|M_1| \ge \cdots \ge |M_u|$. By convention, the value of the empty minor $[\,|\,]$ is 1. The *shape* $|M|$ of M is the sequence $(|M_1|,\ldots,|M_u|)$. A minor $[a_1,\ldots,a_t|b_1,\ldots,b_t]$ of X is a *doset* minor if $\{a_1,\ldots,a_t\} \preceq \{b_1,\ldots,b_t\}$ in H. Let us denote by $\Delta(X)$ the set of all doset minors of X. Let $a_1 =$

$\{a_{11},\ldots,a_{1t_1}\}$, $b_1 = \{b_{11},\ldots,b_{1t_1}\}$, ..., $a_s = \{a_{s1},\ldots,a_{st_s}\}$, $b_s = \{b_{s1},\ldots,b_{st_s}\}$ be elements in H and suppose that $[a_1|b_1],\ldots,[a_s|b_s] \in \Delta(X)$. The product $M = [a_1|b_1]\cdots[a_s|b_s]$ is a *standard* monomial if $b_i \leq a_{i+1}$ in H for all $i = 1,\ldots,s-1$. For a standard monomial M, one denotes by $\min(M)$ the minimum of its factors $[a_1|b_1]$.

The combinatorial structure of $B[X]$ with respect to products of minors is described in the following theorem of De Concini, Eisenbud and Procesi [30, pg.82].

Theorem 1.0.10 *The ring $B[X]$ is a doset algebra on $\Delta(X)$ over B, that is:*
(a) *The standard monomials form a B-basis of $B[X]$.*
(b) (Straightening law) *Let $M_i = [a_{i1},\ldots,a_{ir_i}|b_{i1},\ldots,b_{ir_i}] \in \Delta(X)$ with $i = 1,\ldots,s$ and $N = [c_{11},\ldots,c_{1s_1}|d_{11},\ldots,d_{1s_1}]\cdots[c_{r1},\ldots,c_{rs_r}|d_{r1},\ldots,d_{rs_r}]$ be one of the standard monomials which appear in the (unique) representation of the product $M_1\cdots M_s$ as a linear combination of standard monomials. If one considers $c_i = \{c_{i1},\ldots,c_{is_i}\}$, $d_i = \{d_{i1},\ldots,d_{is_i}\}$, $a_i = \{a_{i1},\ldots,a_{ir_i}\}$ and $b_i = \{b_{i1},\ldots,b_{ir_i}\}$, then in the lexicographic order on H, the sequence $c_1,d_1,\ldots,c_r,d_r$ is less than or equal to every sequence obtained by permuting the elements $a_1,b_1,\ldots,a_s,b_s$.*

The following lemma is a direct consequence of De Concini and Procesi [28, Lemma 5.2]:

Lemma 1.0.11 *Every t-minor $M = [a_1,\ldots,a_t|b_1,\ldots,b_t]$ can be written as a linear combination of doset t-minors. Furthermore, for any t-doset minor $[c_1,\ldots,c_t|d_1,\ldots,d_t]$ which appears in the representation of M we have $c_i \leq a_i$ for all $i = 1,\ldots,t$.*

Set $\Delta(X,\alpha) = \{[a|b] \in \Delta(X) : \alpha \preceq a\}$. By Lemma 1.0.11 one deduces that the ideal $I(X,\alpha)$ is generated by the doset minors in $\Omega(X,\alpha) = \Delta(X)\setminus\Delta(X,\alpha)$, that is, the ideal $I(X,\alpha)$ is the ideal cogenerated by α. Note that the set $\Omega(X,\alpha)$ is the set of all the doset minors whose sequence of row indices is not greater or equal than α. Using the straightening law we get the following:

Corollary 1.0.12 *The set of all standard monomials M such that* $\min(M) \in \Omega(X,\alpha)$ *is a B-basis of $I(X,\alpha)$, and the set of the residue class of all standard monomials M such that* $\min(M) \in \Delta(X,\alpha)$ *forms a B-basis of the ring $R(X,\alpha)$.*

Kutz [53] has shown the following:

Proposition 1.0.13 (a) *The dimension of the ring $R(X,\alpha)$ equals* $\dim B + (n+1)t - \sum_{i=1}^{t}\alpha_i$.
(b) *If B is a Cohen-Macaulay ring, then the ring $R(X,\alpha)$ is Cohen-Macaulay. In particular, the classical ring of symmetric minors $S_{t+1}(X)$ is a Cohen-Macaulay ring of dimension $t(t+1)/2+(n-t)t$.*

A proof of Proposition 1.0.13 will be given later in Chapter 5.

(**A**) Let $X = (X_{ij})$ be an $n \times n$ alternating matrix of indeterminates over a commutative ring B, i.e. $X_{ij} = -X_{ji}$ for all $i > j$ and $X_{ii} = 0$, for short a *generic* alternating matrix, and $B[X] = B[X_{ij} : 1 \le i < j \le n]$ the polynomial ring over B. It is well known that $\det(X) = 0$ when n is odd, while for n even there exists an element $\mathrm{pf}(X)$ in $B[X]$, called the *pfaffian* of X, such that $\det(X) = \mathrm{pf}(X)^2$. For more details about pfaffians we refer the reader to Bourbaki [11], Chapter IX, § 5, no. 2. Let $\Pi(X)$ be the set of all pfaffians of X, and consider $\Pi(X)$ ordered in the following way:

$$[a_1, \ldots, a_{2t}] \preceq [b_1, \ldots, b_{2s}] \text{ if and only if } t \ge s \text{ and } a_i \le b_i \text{ for } i = 1, \ldots, 2s,$$

and let $\pi \in \Pi(X)$, $\pi = [a_1, \ldots, a_{2t}]$. One defines $I(X, \pi)$ to be the ideal generated by all pfaffians which are not greater or equal than π. The ideal $I(X, \pi)$ is called the ideal *cogenerated* by π. Denote by $R(X, \pi)$ the residue class ring of $B[X]$ with respect to the ideal $I(X, \pi)$ and by $\Pi(X, \pi)$ the set of all pfaffians which are greater or equal than π.

In particular, if $\pi = [1, \ldots, 2t]$ then $I(X, \pi)$ is the ideal $I_{t+1}(X)$ generated by all the $(2t+2)$-pfaffians of X, and denote by $P_{t+1}(X)$ the analogous residue class ring. The ideals $I_{t+1}(X)$ are called *pfaffian* ideals and the rings $P_{t+1}(X)$ the *classical ring of pfaffians*, provided B is a field.

Let $\nu = \pi_1 \cdots \pi_u$ be a monomial, i.e. a product of pfaffians. We can always assume that the sizes of pfaffians π_i, denoted by $|\pi_i|$, are in non-increasing order $|\pi_1| \ge \cdots \ge |\pi_u|$. By convention, the value of the empty pfaffian [ ] is 1. The *shape* $|\nu|$ of ν is the sequence $(|\pi_1|, \ldots, |\pi_u|)$. Moreover, the monomial ν is said to be *standard* if and only if $\pi_1 \preceq \cdots \preceq \pi_u$. The combinatorial structure of $B[X]$ with respect to products of pfaffians is settled by the following theorem; see De Concini and Procesi [28, Theorem 6.5] or De Concini, Eisenbud and Procesi [30, pg.53].

Theorem 1.0.14 *The ring $B[X]$ is a graded algebra with straightening law on $\Pi(X)$ over B, that is:*
(a) *The standard monomials form a B-basis of $B[X]$.*
(b) *The product of two pfaffians $\pi_1, \pi_2 \in \Pi(X)$ such that $\pi_1\pi_2$ is not a standard monomial has a representation*

$$\pi_1\pi_2 = \sum \lambda_i \xi_i \eta_i, \quad \lambda_i \in B, \quad \lambda_i \ne 0,$$

where $\xi_i\eta_i$ is a standard monomial and $\xi_i \prec \pi_1$, $\pi_2 \prec \eta_i$ (we allow here that η_i is the empty pfaffian).
(c) *The standard representation of an arbitrary monomial ν can be found by successive applications of the straightening relations in* (b).

The straighten of two pfaffians relies on the following result of De Concini and Procesi [28, Lemma 6.1]:

Lemma 1.0.15 *We have*

$$[a_1,\dots,a_{2t}][b_1,\dots,b_{2s}] - \sum_{i=1}^{2t}[a_1,\dots,a_{i-1},b_1,a_{i+1},\dots,a_{2t}][a_i,b_2,\dots,b_{2s}] = \sum_{j=2}^{2s}(-1)^{j-1}[b_2,\dots,b_{j-1},b_{j+1},\dots,b_{2s}][b_j,b_1,a_1,\dots,a_{2t}].$$

Note that Lemma 1.0.15 yields an argument for part (b) of Theorem 1.0.14, as long as any straightening relation can be obtained by iterated use of Lemma 1.0.15.

As a consequence of Theorem 1.0.14 and Proposition 1.0.2 we get:

Corollary 1.0.16 *The ring $R(X,\pi)$ is a graded algebra with straightening law on $\Pi(X,\pi)$ over B.*

The sets $\Pi(X)$ and $\Pi(X,\pi)$ are distributive lattices. By virtue of Bruns and Vetter [21, Theorem (5.14)] we can immediately conclude:

Proposition 1.0.17 *If B is a Cohen-Macaulay ring, then $R(X,\pi)$ is Cohen-Macaulay.*

In the next proposition we compute the dimension of the ring $R(X,\pi)$ with $\pi=[a_1,\dots,a_{2t}]\in\Pi(X)$.

Proposition 1.0.18 *We have* $\dim R(X,\pi)=\dim B+2nt-\sum_{i=1}^{2t}a_i$. *In particular, the classical ring of pfaffians $P_{t+1}(X)$ is Cohen-Macaulay of dimension $2nt-(2t+1)t$.*

Proof. Since the poset $\Pi(X,\pi)$ is a distributive lattice, it is a wonderful poset, so that any two maximal chains have the same length. The rank of $\Pi(X,\pi)$ is the length of every maximal chain in $\Pi(X)$ which starts from π. We can find the following saturated chain $[a_1,\dots,a_{2t-1},a_{2t}]<[a_1,\dots,a_{2t-1},a_{2t}+1]<\dots<[a_1,\dots,a_{2t-1},n]<\dots<[a_1,\dots,n-1,n]<[a_1,\dots,a_{2t-2}]$. By induction on t we get that $\operatorname{rank}\Pi(X,\pi)=2n-(a_{2t-1}+a_{2t})+2n(t-1)-\sum_{i=1}^{2t-2}a_i-1$, hence $\operatorname{rank}\Pi(X,\pi)=2nt-\sum_{i=1}^{2t}a_i-1$. As $R(X,\pi)$ is an ASL over B we can compute its dimension by using Bruns and Vetter[21, Prop. (5.10)], so that $\dim R(X,\pi)=\dim B+\operatorname{rank}\Pi(X,\pi)+1$. □

Recall that a finitely generated module M over a Noetherian ring R is called *perfect* if grade (M) = projdim (M). In particular, an ideal I is perfect if grade (I) = projdim (R/I). We now get that the ideals $I(X,\pi)$, $\pi\in\Pi(X)$, are perfect ideals.

Proposition 1.0.19 *If B is a Noetherian ring, then the ideal $I(X,\pi)$ is perfect of grade $n(n-1)/2 - 2nt + \sum_{i=1}^{2t} a_i$.*

Proof. To begin with let us consider $B = \mathbb{Z}$ and show that the ideal $I(X,\pi)$ is perfect of grade $n(n-1)/2 - 2nt + \sum_{i=1}^{2t} a_i$. As $R(X,\pi)$ is an ASL on $\Pi(X,\pi)$ over $\mathbb{Z}$, it is a free $\mathbb{Z}$-module. Consequently, it is enough to show that $R(X,\pi)$ is a perfect $\mathbb{Z}_p[X]$-module for all primes $p \in \mathbb{Z}$. We have that $R(X,\pi)$ is a Cohen-Macaulay $\mathbb{Z}_p[X]$-module (see Proposition 1.0.17), $\operatorname{projdim}_{\mathbb{Z}_p[X]}(R(X,\pi))$ is finite, and $R(X,\pi)$ has no non-trivial idempotents (the idempotents of $R(X,\pi)$ must be of degree zero). Applying Bruns and Vetter [21, Prop. (16.19)] we get that $R(X,\pi)$ is a perfect $\mathbb{Z}_p[X]$-module of grade equal to $\operatorname{grade} I(X,\pi) = \operatorname{height} I(X,\pi) = \dim \mathbb{Z}_p[X] - \dim R(X,\pi) = n(n-1)/2 - 2nt + \sum_{i=1}^{2t} a_i$. □

Chapter 2

Divisor class group and canonical class of determinantal rings

2.1 Integrity and normality

(**G**) Bruns and Vetter [21, Theorem (6.3)] have proved the following:

Theorem 2.1.1 *Let B be a (normal) domain. Then the ring $R(X,\delta)$ is a (normal) domain.*

Theorem 2.1.1 has an obvious consequence.

Corollary 2.1.2 *Let B be an integral domain, and $\Omega \subset \Delta(X)$ a poset ideal. Then the minimal prime ideals of $\Omega R(X,\delta)$ are the ideals*

$$I(x,\zeta) = I(X,\zeta)/I(X,\delta),$$

ζ running through the minimal elements of $\Delta(X)\setminus\Omega$, and $\Omega R(X,\delta)$ is their intersection.

(**S**) We prove that $R(X,\alpha)$ is a (normal) domain as long as B is a (normal) domain. In order to do this, we need the following lemma which describes the localization of $R(X,\alpha)$ with respect to α within the total ring of quotients of $R(X,\alpha)$. The proof will follow closely the argument of [21, Lemma (6.4)].

Let $\alpha \in H$, $\alpha = \{\alpha_1,\dots,\alpha_t\}$, and denote by f the residue class of the minor $[\alpha|\alpha]$ in $R(X,\alpha)$. Consider $\Psi = \{[\alpha_i|\alpha_j] : 1 \le i < j \le t\} \cup \{[\alpha|\beta] \in \Delta(X,\alpha) : \beta$ differs from α in exactly one index$\}$ and denote by $B[\Psi]$ the B-subalgebra of $R(X,\alpha)$ generated by the elements in set Ψ.

Lemma 2.1.3 *The set Ψ is algebraically independent over B and*

$$R(X,\alpha)[f^{-1}] = B[\Psi][f^{-1}].$$

Thus $R(X,\alpha)[f^{-1}]$ is isomorphic to $B[T_1,\ldots,T_d][\zeta^{-1}]$, where $\zeta \in B[T_1,\ldots,T_d]$ and $d = \dim R(X,\alpha) - \dim B$. If B is an integral domain, then ζ is a prime element.

Proof. First we prove that $R(X,\alpha)[f^{-1}] = B[\Psi][f^{-1}]$. It is clear that $B[\Psi][f^{-1}]$ is contained in $R(X,\alpha)[f^{-1}]$. The other inclusion will be done if we show that $[u|v] \in B[\Psi][f^{-1}]$ for all $[u|v] \in \Delta(X,\alpha)$. If u (or v) is in $\{\alpha_1,\ldots,\alpha_t\}$, then we get $[u,\alpha_1,\ldots,\alpha_t|v,\alpha_1,\ldots,\alpha_t] = 0$. Expanding the minor with respect to the first row we get

$$[u|v]f = \sum_{j=1}^{t}(-1)^{j+1}[u|\alpha_j][\alpha_1,\ldots,\alpha_t|v,\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_t],$$

where $\hat{w}$ means that the index w is missing. Because $[u|\alpha_j] \in \Psi$ and

$$[\alpha_1,\ldots,\alpha_t|v,\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_t] \in \Psi$$

(it differs from α in exactly one index) or $[\alpha_1,\ldots,\alpha_t|v,\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_t] = 0$ in $R(X,\alpha)$, we conclude that $[u|v] \in B[\Psi][f^{-1}]$. The general case is an immediate consequence of the relation above.

Since f is the only minimal element of $\Delta(X,\alpha)$, it is a non zero-divisor in $R(X,\alpha)$, therefore

$$\dim B[\Psi] = \dim B[\Psi][f^{-1}] = \dim R(X,\alpha)[f^{-1}] = \dim R(X,\alpha).$$

Note that an element $\xi \in \Delta(X,\alpha)$ which differs from α in exactly one index must be a t-minor, so that $|\Psi| = t(t+1) + \sum_{i=1}^{t}(n-\alpha_i-(t-i)) = t(n+1) - \sum_{i=1}^{t}\alpha_i = \dim R(X,\alpha) - \dim B$. Consequently, when B is a field the set Ψ is algebraically independent over B. The general case proceeds as in the proof of Lemma (6.1) in [21].

When B is an integral domain, the element f is a prime element of the ring $B[(X_{a_ia_j})_{1\le i\le j\le t}]$ because it is the determinant of the symmetric matrix $(X_{a_ia_j})_{1\le i\le j\le t}$ (if we use Proposition 2.1.9, it is immediately that the determinant of a symmetric matrix is a prime element). Therefore f is a prime element of $B[\Psi]$, and the isomorphism above maps f in ζ. We get that ζ is a prime element of $B[T_1,\ldots,T_d]$. □

Corollary 2.1.4 *Let B be a domain. Then the ring $R(X,\alpha)$ is a domain.*

Proof. The localization $R(X,\alpha)[f^{-1}]$ is a domain, by Lemma 2.1.3, and f is a non zero-divisor. Therefore $R(X,\alpha)$ is a domain, too. □

In order to prove normality of the ring $R(X,\alpha)$ we shall use Serre's normality criterion [37, Theorem 4.1]. Since it is Cohen-Macaulay 1.0.13, it is enough to show that $R(X,\alpha)$ satisfies the Serre's condition (R_1). By 2.1.3, any localization of $R(X,\alpha)$ to a prime which does not contain f is regular, as being a localization of a polynomial ring. Consequently, we have to investigate the localizations of $R(X,\alpha)$ to minimal prime ideals of f.

First let us describe the minimal prime ideals of f. Set $U = \{\beta \in H : \beta > \alpha\}$, and denote by J the set of the minimal elements in U, that is J is the set of the upper neighbours of α in H. One considers $J = \emptyset$ if $\alpha = \{n\}$, and $I(X,\emptyset) = (X_{ij} : 1 \le i \le j \le n)$. For $\beta \in J$, one has $I(X,\alpha) \subset I(X,\beta)$.

Lemma 2.1.5 *One has* $\bigcap_{\beta\in J} I(X,\beta) = ([\alpha|\gamma] : [\alpha|\gamma] \in \Delta(X,\alpha)) + I(X,\alpha)$

Proof. It is obvious that the right side is contained in the left side. Let $\mu \in ([\alpha|\gamma] : [\alpha|\gamma] \in \Delta(X,\alpha)) + I(X,\alpha)$. One may assume that μ is a standard monomial with $\min(\mu) = [\gamma|\delta]$. Since $\gamma \not\geq \beta$ for all $\beta \in J$, one deduces that $\gamma = \alpha$ or $\gamma \not\geq \alpha$ and the proof is done. □

Lemma 2.1.6 *The ideals* $I(x,\beta) = I(X,\beta)/I(X,\alpha)$ *of* $R(X,\alpha)$, *with* $\beta \in J$, *are the minimal prime ideals of* (f).

Proof. Note that the ideals $I(x,\beta)$, $\beta \in J$, are distinct. Furthermore, we have

$$(\bigcap_{\beta\in J} I(x,\beta))^2 \subset (f).$$

By Lemma 2.1.5, it is enough to show that for all $\gamma,\delta > \alpha$ we have $[\alpha|\gamma][\alpha|\beta] \in (f)$. Let ν a standard monomial that appear in the standard representation of $[\alpha|\gamma][\alpha|\beta]$, and set $\min(\nu) = [\zeta_1|\zeta_2]$. The sequence ζ_1,ζ_2 is less than or equal to the sequence α,α, hence $\nu = [\alpha|\alpha]\nu_1$ or $\nu = 0$ in $R(X,\alpha)$. □

We want now to describe the set J of all the upper neighbours of α. The standard way to do this is to break α into blocks of consecutive integers

$$\alpha = [\beta_1,\dots,\beta_r], \qquad \beta_i = (\alpha_{k_{i-1}+1},\dots,\alpha_{k_i}),$$

where $k_0 = 0$ and $k_r = t$. Each β_i is followed by a gap

$$\chi_i = (\alpha_{k_i}+1,\dots,\alpha_{k_i+1}-1),$$

the sequence of integers properly between the last element of β_i and the first element of β_{i+1}, and $\chi_r = (\alpha_t+1,\dots,n)$ or $\chi_r = \emptyset$ if $\alpha_t = n$. For every $i = 1,\dots,r$ we get an upper neighbour ζ_i of α by replacing α_{k_i} with $\alpha_{k_i}+1$, when $i < r$ or $i = r$ and $\alpha_t < n$. If $\alpha_t = n$, then ζ_r is obtained from α by deleting α_t.

The following lemma, borrowed from [49, Lemma (1.1)], will allow us inductively proofs.

Lemma 2.1.7 *Let $\tilde{X}$ be a symmetric matrix of size $(n-1)\times(n-1)$. Consider the K-algebras homomorphism*

$$\psi : K[X][X_{11}^{-1}] \longrightarrow K[\tilde{X}][X_{11},\dots,X_{1n}][X_{11}^{-1}],$$

defined by the assignment $\psi(X_{1j}) = X_{1j}$ for all $j = 1,\dots,n$, and $\psi(X_{ij}) = \tilde{X}_{i-1j-1} + X_{1i}X_{1j}X_{11}^{-1}$ for all $1 < i \le j \le n$. Then ψ is an isomorphism.

Let us denote by $[\dots|\dots]_X$, and by $[\dots|\dots]_{\tilde{X}}$ the minors of X and $\tilde{X}$, respectively. Let $\mu = [a_1,\dots,a_s|b_1,\dots,b_s]_X \in \Delta(X)$. Since the matrix $(X_{1i}X_{1j}X_{11}^{-1})_{1\le i,j\le n}$ has the rank 1, and using the linearity of the determinant with respect to the rows one obtains

$$\psi(\mu) = [a_1-1,\dots,a_s-1|b_1-1,\dots,b_s-1]_{\tilde{X}} + \sum_{u,v=1}^{s}(-1)^{u+v}X_{1b_v}X_{1a_u}X_{11}^{-1}\mu_{uv}, \tag{2.1}$$

where $\mu_{uv} = [a_1-1,\dots,\widehat{a_u-1},\dots,a_s-1|b_1-1,\dots,\widehat{b_v-1},\dots,b_s-1]_{\tilde{X}}$.

In particular one gets

$$\psi([1,a_2,\dots,a_s|1,b_2,\dots,b_s]_X) = X_{11}[a_2-1,\dots,a_s-1|b_2-1,\dots,b_s-1]_{\tilde{X}}. \tag{2.2}$$

If $\alpha_1 = 1$, then $X_{11} \notin I(X,\alpha)$, and using (2.1) and (2.2) one shows that the ideal $\psi(I(X,\alpha))$ is the extension of the ideal $I(\tilde{X},\tilde{\alpha})$ to the ring $K[\tilde{X}][X_{11},\dots,X_{1n}][X_{11}^{-1}]$, where $\tilde{\alpha} = \{\alpha_2-1,\dots,\alpha_t-1\}$. Thus we get an isomorphism

$$\tilde{\psi} : R(X,\alpha)[x_{11}^{-1}] \longrightarrow R(\tilde{X},\tilde{\alpha})[X_{11},\dots,X_{1n}][X_{11}^{-1}], \tag{2.3}$$

where x_{11} denotes the residue class of X_{11} in $R(X,\alpha)$. If one denotes by $\tilde{f}$ the residue class in $R(\tilde{X},\tilde{\alpha})$ of the minor $[\tilde{\alpha}|\tilde{\alpha}]$, by $\tilde{\zeta_i}$ the upper neighbours of $\tilde{\alpha}$, and by $I(\tilde{x},\tilde{\zeta_i})$ the corresponding minimal prime ideals of $\tilde{f}$, then by (2.2) we have $\tilde{\psi}(f) = X_{11}\tilde{f}$. Moreover, if $\alpha_2 = 2$, then

$$\tilde{\psi}(I(x,\zeta_i)) = I(\tilde{x},\tilde{\zeta_i})R(\tilde{X},\tilde{\alpha})[X_{11},\dots,X_{1n}][X_{11}^{-1}], \tag{2.4}$$

for all $i = 1,\dots,r$, and, if $\alpha_2 > 2$, then

$$\tilde{\psi}(I(x,\zeta_i)) = I(\tilde{x},\tilde{\zeta}_{i-1})R(\tilde{X},\tilde{\alpha})[X_{11},\ldots,X_{1n}][X_{11}^{-1}], \tag{2.5}$$

for all $i = 1,\ldots,r$, while $I(x,\zeta_1)R(X,\alpha)[x_{11}^{-1}] = R(X,\alpha)[x_{11}^{-1}]$.

Set $R = R(X,\alpha)$, and $P_i = I(x,\beta_i)$. We are ready now to prove that the localizations R_{P_i} are discrete valuation rings (for short DVR).

Lemma 2.1.8 (a) *R_{P_i} is a regular ring for all $i = 1,\ldots,r$. Hence R_{P_i} is a DVR, and let v_i be the corresponding valuation.*
(b) *If $i < r$, or if $i = r$ and $\alpha_t < n$, then $v_i(f) = 2$. If $\alpha_t = n$, then $v_r(f) = 1$.*

Proof. The proof goes by induction on n. For $n = 2$, α is one of the following $\{1\}$, $\{2\}$, $\{1,2\}$.

If $\alpha = \{1\}$, then $I(X,\alpha) = (X_{11}X_{22} - X_{12}^2)$, $f = x_{11}$, and $P_1 = (x_{11},x_{12})$. We get that $P_1R_{P_1} = (x_{12})R_{P_1}$, and $f = x_{22}^{-1}x_{12}^2$, therefore R_{P_1} is regular and $v_1(f) = 2$.

If $\alpha = \{2\}$, or $\alpha = \{1,2\}$, then $I(X,\alpha) = (X_{11},X_{12})$, $f = x_{22}$, and $P_1 = (x_{22})$, or $I(X,\alpha) = (0)$, $f = (X_{11}X_{22} - X_{12}^2)$, and $P_1 = (X_{11}X_{22} - X_{12}^2)$. In this case all the assertions are trivial.

Let us suppose now $n > 2$. If $\alpha_1 > 1$, Then $R(X,\alpha) \cong R(Y,\gamma)$, where Y is an $(n+1-\alpha_1)\times(n+1-\alpha_1)$ symmetric matrix of indeterminates and $\gamma = \{1,\alpha_2+1-\alpha_1,\ldots,\alpha_t+1-\alpha_1\}$. By induction, we may assume $\alpha_1 = 1$. If $\alpha_2 = 2$, then R_{P_i} is localization of $R[x_{11}^{-1}]$ for all $i = 1,\ldots,r$. Set $\tilde{R} = R(\tilde{X},\tilde{\alpha})$, and $\tilde{P}_i = I(\tilde{x},\tilde{\beta}_i)$. From (2.3) and (2.4) we get that R_{P_i} is isomorphic to a localization of $\tilde{R}_{\tilde{P}_i}[X_{11},\ldots,X_{1n}]$. It is a regular ring by induction hypothesis, hence R_{P_i} is regular. Since X_{11} is an unit, we have $v_i(f) = \tilde{v}_i(\tilde{f})$, where $\tilde{v}_i$ is the valuation on $\tilde{R}_{\tilde{P}_i}$. Again by induction $\tilde{v}_r(\tilde{f}) = 1$ if $\alpha_t - 1 = n-1$, and $\tilde{v}_i(\tilde{f}) = 2$ in the other cases. If $\alpha_2 > 2$ and $i > 1$, one proceeds as above using (2.3) and (2.5).

It remains the case $i = 1$ and $\alpha_2 > 2$. By definition $P_1 = (x_{11},\ldots,x_{1n})$. Since all the 2-minors of the first two rows of X are 0 in R, we get $x_{1i} = x_{12}x_{i2}x_{22}^{-1}$ in R_{P_i}, and $P_1R_{P_1} = (x_{12})R_{P_1}$. Therefore R_{P_1} is regular. In R_{P_1} we have that $f = \det(x_{ij})_{1\le i,j\le \alpha_t} = [\zeta_1|\zeta_1]x_{22}^{-2}x_{12}^2$. As the element $[\zeta_1|\zeta_1]x_{22}^{-2}$ is invertible in R_{P_1}, one gets $v_1(f) = 2$. □

An immediate application of Serre's normality criterion [37, Theorem 4.1] gives us the following

Theorem 2.1.9 *Let B be a normal domain. Then the ring $R(X,\alpha)$ is also a normal domain.*

Proof. Since $R(X,\alpha)$ is Cohen-Macaulay 1.0.13, it is enough to show that $R(X,\alpha)$ satisfies the Serre's condition (R_1). By 2.1.3, any localization of

$R(X,\alpha)$ to a prime which does not contain f is a localization of a polynomial ring, hence a regular ring. By Lemma 2.1.8(a), every localization of $R(X,\alpha)$ to a minimal prime ideal of f is regular. Consequently the ring $R(X,\alpha)$ satisfies the Serre's condition (R_1). □

(**A**) In this part, as a normality criterion we shall use the following

Lemma 2.1.10 *Let S be a Noetherian ring, and $x \in S$ a non zero-divisor. Assume $S[x^{-1}]$ is a normal domain, and $S/(x)$ is reduced. Then S is normal.*

Proof. See [19, Exercise 2.2.33] or [21, Lemma (16.24)]. □

In order to prove that $R(X,\pi)$ is a (normal) domain as long as B is a (normal) domain, we need the following lemma which describes the localization of $R(X,\pi)$ with respect to π within the total ring of quotients of $R(X,\pi)$. Our argument is a suitable modification of the argument of [21, Lemma (6.4)].

Let $\pi \in \Pi(X)$, $\pi = [a_1,\dots,a_{2t}] \in \Pi(X)$, $\Psi = \{[a_i a_j] : 1 \le i < j \le 2t\} \cup \{\xi \in \Pi(X,\pi) : \xi$ differs from π in exactly one index$\}$ and denote by $B[\Psi]$ the B-subalgebra of $R(X,\pi)$ generated by the elements in set Ψ.

Lemma 2.1.11 *The set Ψ is algebraically independent over B and*

$$R(X,\pi)[\pi^{-1}] = B[\Psi][\pi^{-1}].$$

Thus $R(X,\pi)[\pi^{-1}]$ is isomorphic to $B[T_1,\dots,T_d][\zeta^{-1}]$, where $\zeta \in B[T_1,\dots,T_d]$ and $d = \dim R(X,\pi) - \dim B$. If B is an integral domain, then ζ is a prime element.

Proof. At the beginning we prove that $R(X,\pi)[\pi^{-1}] = B[\Psi][\pi^{-1}]$. It is clear that $B[\Psi][\pi^{-1}]$ is contained in $R(X,\pi)[\pi^{-1}]$. The other inclusion will be done if we show that $[uv] \in B[\Psi][\pi^{-1}]$ for all $[uv] \in \Pi(X,\pi)$. If u (or v) is in $\{a_1,\dots,a_{2t}\}$, then $[u,v,a_1,\dots,a_{2t}] = 0$. Expanding the pfaffian with respect to the first two rows we get

$$[uv]\pi = \sum_{j=1}^{2t} (-1)^{j+1}[ua_j][v,a_1,\dots,\hat{a}_j,\dots,a_{2t}],$$

where $\hat{w}$ means that the index w is missing. Because $[ua_j] \in \Psi$ and

$$[v,a_1,\dots,\hat{a}_j,\dots,a_{2t}] \in \Psi$$

(it differs from π in exactly one index) or $[v,a_1,\dots,\hat{a}_j,\dots,a_{2t}] = 0$ in $R(X,\pi)$, we conclude that $[uv] \in B[\Psi][\pi^{-1}]$. The general case is an immediate consequence of the relation above.

Since π is the only minimal element of $\Pi(X,\pi)$, it is not a zero-divisor in $R(X,\pi)$, therefore

$$\dim B[\Psi] = \dim B[\Psi][\pi^{-1}] = \dim R(X,\pi)[\pi^{-1}] = \dim R(X,\pi).$$

Note that an element $\xi \in \Pi(X,\pi)$ which differs from π in exactly one index must be a $2t$-pfaffian, so that $|\Psi| = t(2t-1) + \sum_{i=1}^{2t}(n - a_i - (2t - i)) = 2nt - \sum_{i=1}^{2t} a_i = \dim R(X,\pi) - \dim B$. Consequently, when B is a field the set Ψ is algebraically independent over B. The general case proceeds as in the proof of Lemma (6.1) in [21].

When B is an integral domain, the element π is a prime element of the ring $B[(X_{a_i a_j})_{1\le i<j\le 2t}]$ because it is the pfaffian of the antisymmetric matrix $(X_{a_i a_j})_{1\le i\le j\le 2t}$ (if we use 2.1.12, it is immediately that the pfaffian of an antisymmetric matrix of even dimension is a prime element). Therefore π is a prime element of $B[\Psi]$, and taking into account that the isomorphism above maps π in ζ we get that ζ is a prime element of $B[T_1,\ldots,T_d]$. □

Theorem 2.1.12 *Let B be a (normal) domain. Then the ring $R(X,\pi)$ is a (normal) domain.*

Proof. By 2.1.11, the ring $R(X,\pi)[\pi^{-1}]$ is a domain. As we already observed, π is a non zero-divisor in $R(X,\pi)$, therefore $R(X,\pi)$ is contained in $R(X,\pi)[\pi^{-1}]$. For normal B the ring $B[\Psi][\pi^{-1}]$ is also normal, hence $R(X,\pi)[\pi^{-1}]$ is normal. As the ring $R(X,\pi)/\pi R(X,\pi)$ is a graded ASL over a domain, then by [21, Prop. (5.7)] it is reduced and we can apply 2.1.10. □

Corollary 2.1.13 *Let B be an integral domain, and $\Omega \subset \Pi(X,\pi)$ a poset ideal. Then the minimal prime ideals of $\Omega R(X,\pi)$ are the ideals*

$$I(x,\zeta) = I(X,\zeta)/I(X,\pi),$$

ζ running through the minimal elements of $\Pi(X,\pi)\setminus\Omega$, and $\Omega R(X,\pi)$ is their intersection.

Proof. Let ζ be a minimal element of π. The factor ring $R(X,\pi)/I(x,\zeta)$ is isomorphic to $R(X,\zeta)$ and this is a domain by 2.1.12, so that $I(x,\zeta)$ is a prime ideal. Because Ω is a poset ideal, we get that $\Omega \cup (\Pi(X)\setminus\Pi(X,\pi)) = \bigcap(\Pi(X)\setminus\Pi(X,\zeta))$. Now we get that $\Omega R(X,\pi) = \bigcap I(x,\zeta)$, ζ running through the minimal elements of $\Pi(X,\pi)\setminus\Omega$. From this equation it follows that these are all the minimal prime ideals of $\Omega R(X,\pi)$. □

2.2 Divisor class group of determinantal rings

(**G**) The divisor class group $\mathrm{Cl}(R(X,\delta))$ of determinantal rings $R(X,\delta)$, was determined by Bruns and Vetter [21, Theorem (8.3)].

Theorem 2.2.1 *Let B be a Noetherian normal domain, X an $m \times n$ generic matrix over B, and $\delta = [a_1, \ldots, a_t | b_1, \ldots, b_t] \in \Delta(X)$. Then the divisor class group of $R(X,\delta)$ is given by*

$$\mathrm{Cl}(R(X,\delta)) = \begin{cases} \mathrm{Cl}(B) \oplus \mathbb{Z}^{u+v} & \text{if } a_t = m \text{ or } b_t = n, \\ \mathrm{Cl}(B) \oplus \mathbb{Z}^{u+v+1} & \text{otherwise,} \end{cases}$$

where $u+1$, resp. $v+1$ represents the number of blocks (of consecutive integers) for the row part $[a_1, \ldots, a_t]$, resp. column part $[b_1, \ldots, b_t]$.

As a by-product we get that the ring $R(X,\delta)$ is factorial if and only if it is a polynomial ring over B.

Corollary 2.2.2 *The ring $R(X,\delta)$ is factorial if and only if B is factorial and*

$$\delta = [a_1, \ldots, a_1+t-1 | n-t+1, \ldots, n] \text{ or } \delta = [m-t+1, \ldots, m | b_1, \ldots, b_1+t-1].$$

(**S**) For the computation of the divisor class group of $R(X,\alpha)$ we use the classical theorem of Nagata which relates the divisor class group of a Noetherian normal domain and the divisor class group of its rings of quotients (see [37, §7] or [7, §4.5]).

Theorem 2.2.3 *Let B be a Noetherian normal domain, X an $n \times n$ generic symmetric matrix over B, and $\alpha = \{\alpha_1, \ldots, \alpha_t\} \in H$. The divisor class group of $R(X,\alpha)$ is generated by the classes of the prime ideals $I(x,\zeta_i)$, $1 \le i \le r$, the only relation between them is $\sum_{i=1}^{r} v_i(f)\mathrm{cl}(I(x,\zeta_i)) = 0$, and*

$$\mathrm{Cl}(R(X,\alpha)) = \begin{cases} \mathrm{Cl}(B) \oplus \mathbb{Z}^{r-1} \oplus \mathbb{Z}_2 & \text{if } \alpha_t < n \\ \mathrm{Cl}(B) \oplus \mathbb{Z}^{r-1} & \text{if } \alpha_t = n. \end{cases}$$

Proof. One can first assume that B is a field. By Nagata's theorem and Lemma 2.1.3 we deduce that $\mathrm{Cl}(R(X,\alpha))$ is generated by the classes of the minimal prime ideals of f, that is $\mathrm{cl}(I(x,\zeta_i))$, $1 \le i \le r$. Since the ring $R(X,\alpha)[f^{-1}]$ arises from a polynomial ring by inversion of a prime element (see 2.1.3), we get that the units of $R(X,\alpha)[f^{-1}]$ are elements of the form af^m, with $a \in K$, $a \neq 0$ and $m \in \mathbb{Z}$. A similar argument to [21, pg.94] shows that the only relation between $\mathrm{cl}(I(x,\zeta_1)), \ldots, \mathrm{cl}(I(x,\zeta_r))$ is $\sum_{i=1}^{r} v_i(f)\mathrm{cl}(I(x,\zeta_i)) = 0$. The description of the divisor class group of $R(X,\alpha)$ follows readily from 2.1.8(b). □

We can now determine when the ring $R(X,\alpha)$ is factorial.

Corollary 2.2.4 *The ring $R(X,\alpha)$ is factorial if and only if B is factorial and $\alpha = \{n-t+1,\dots,n\}$.*

Proof. The ring $R(X,\alpha)$ is factorial if and only if $\mathrm{Cl}(R(X,\alpha)) = 0$. This condition is fulfilled whether $\mathrm{Cl}(B) = 0$ and $r = 1$ in the second case. This means that B is factorial and $\alpha = \{n-t+1,\dots,n\}$. □

Actually, when $\alpha = \{n-t+1,\dots,n\}$ the ring $R(X,\alpha)$ is isomorphic to a polynomial ring over B.

(**A**) The computation of the divisor class group of ring $R(X,\pi)$ also involve the use of Nagata's theorem. Let B be a Noetherian normal domain, and $\pi = [a_1,\dots,a_{2t}] \in \Pi(X)$. As we already know, if B is a Noetherian normal domain then $R(X,\pi)$ is also a Noetherian normal domain; see Theorem 2.1.12. Under these assumptions, Bruns and Vetter [21, Chapter 8] have shown that

$$\mathrm{Cl}(R(X,\pi)) = \mathrm{Cl}(B) \oplus \mathrm{Ker}\,(\mathrm{Cl}(R(X,\pi)) \longrightarrow \mathrm{Cl}(R(X,\pi)[\pi^{-1}])).$$

By Nagata's theorem we get that $\mathrm{Ker}\,(\mathrm{Cl}(R(X,\pi)) \longrightarrow \mathrm{Cl}(R(X,\pi)[\pi^{-1}]))$ is generated by the classes of minimal prime ideals of $\pi R(X,\pi)$. From 2.1.13 it follows that the ideals $I(x,\zeta)$, ζ running through the upper neighbours of π, are the minimal prime ideals of $\pi R(X,\pi)$. Let us denote by ζ_i, $0 \le i \le u$ the upper neighbours of π. Then

$$\textstyle\sum_{i=0}^{u} \mathrm{cl}(I(x,\zeta_i)) = 0,$$

and every subset of u of them are linearly independent over $\mathbb{Z}$; see [21, pg.94]. All we have to do now is to find out the upper neighbours of π.

The standard way to describe the upper neighbours of π is to break it into blocks of consecutive integers

$$\pi = [\beta_o,\dots,\beta_s], \qquad \beta_i = (a_{k_i+1},\dots,a_{k_{i+1}}),$$

where $k_0 = 0$ and $k_{s+1} = 2t$. Each β_i is followed by the gap

$$\chi_i = (a_{k_{i+1}}+1,\dots,a_{k_{i+1}+1}-1),$$

the sequence of integers properly between the last element of β_i and the first element of β_{i+1}, and $\chi_s = (a_{2t}+1,\dots,n)$, $\chi_s = \emptyset$ if $a_{2t} = n$. For every $i = 0,\dots,u$ we get an upper neighbour of π by replacing $a_{k_{i+1}}$ with $a_{k_{i+1}}+1$, where $u = s$ if $a_{2t} < n$ and $u = s-1$ if $a_{2t} = n$. If $a_{2t} = n$ and $a_{2t-1} = n-1$ we can get another upper neighbour of π, namely

$$\zeta = [a_1,\dots,a_{2t-2}].$$

(In fact, ζ is an upper neighbour of π if and only if $a_{2t} = n$ and $a_{2t-1} = n-1$.)

We are now ready to compute the divisor class group of $R(X,\pi)$.

Theorem 2.2.5 *Let B be a Noetherian normal domain, X an $n \times n$ generic alternating matrix over B, and $\pi = [a_1, \dots, a_{2t}] \in \Pi(X)$. Then the divisor class group of $R(X, \pi)$ is given by*

$$\mathrm{Cl}(R(X,\pi)) = \begin{cases} \mathrm{Cl}(B) \oplus \mathbb{Z}^s & \text{if } a_{2t} < n \text{ or } a_{2t} = n \text{ and } a_{2t-1} = n-1 \\ \mathrm{Cl}(B) \oplus \mathbb{Z}^{s-1} & \text{if } a_{2t} = n \text{ and } a_{2t-1} < n-1. \end{cases}$$

The summand $\mathrm{Cl}(B)$ arises naturally from the embedding $B \longrightarrow R(X, \pi)$, and the summand $\mathbb{Z}^s$ or $\mathbb{Z}^{s-1}$ is generated by the classes of any set of s or $s-1$, respectively of the prime ideals $I(x, \zeta_i)$ and $I(x, \zeta)$.

We can now determine when the rings under consideration are factorial.

Corollary 2.2.6 *The ring $R(X, \pi)$ is factorial if and only if B is factorial and*

$$\pi = [a_1, \dots, a_1 + 2t - 1] \text{ or } \pi = [a_1, \dots, a_1 + 2t - 2, n].$$

Proof. The ring $R(X, \pi)$ is factorial if and only if $\mathrm{Cl}(R(X, \pi)) = 0$. This condition is fulfilled whether $\mathrm{Cl}(B) = 0$ and $s = 0$ in the first case, or $s = 1$ in the second case. This means that B is factorial and $a_1, \dots, a_{2t}$ in the first case, respectively $a_1, \dots, a_{2t-1}$ in the second case are consecutive integers. □

Remark 2.2.7 If $\pi = [a_1, \dots, a_1 + 2t - 1]$, then the ring $R(X, \pi)$ is isomorphic to $B[Y]/I_{t+1}(Y)$, and if $\pi = [a_1, \dots, a_1 + 2t - 2, n]$ then $R(X, \pi)$ is isomorphic to a polynomial ring over $B[Y]/I_t(Y)$, where Y is a generic alternating matrix suitably chosen in each of the two cases. This means that the pfaffian rings $R(X, \pi)$ are factorial only in the classical case. The result for the classical ring of pfaffians $P_{t+1}(X)$ was settled by Avramov [3].

2.3 The canonical class of determinantal rings

Given a normal Cohen-Macaulay domain S with a canonical module ω_S, it is known that ω_S is a rank 1 reflexive module, therefore ω_S is isomorphic to a divisorial ideal of S; see Bruns and Vetter [19, Cor.3.3.19]. Its class in $\mathrm{Cl}(S)$ is called a *canonical class*. Furthermore, if S is a positively graded algebra over a field, then S admits a canonical module and this is unique up to isomorphism, consequently there exists a unique canonical class of S. If $B = K$ is a field, then the rings $R(X, \delta)$, $R(X, \alpha)$, and $R(X, \pi)$ are normal Cohen-Macaulay domains and positively graded K-algebras.

(G) Let $\delta = [a_1, \dots, a_t | b_1, \dots, b_t] \in \Delta(X)$, ζ_i and η_i the upper neighbours of δ arising from raising a row, resp. column index, and in case $a_t = m$, $b_t = n$, $\gamma = [a_1, \dots, a_{t-1} | b_1, \dots, b_{t-1}]$. Denote by $u + 1$, resp. $v + 1$ the number of blocks (of consecutive integers) for the row part $[a_1, \dots, a_t]$, resp. column part $[b_1, \dots, b_t]$. Furthermore let $w = u$ if $a_t < m$, $w = u - 1$ if $a_t = m$, and $z = v$ if $b_t < n$, $z = v - 1$ if $b_t = n$. With these notation we can assert:

Theorem 2.3.1 *Let B be a normal Cohen-Macaulay domain which admits a canonical module ω_B, and let $\mathrm{cl}(\omega)$ be a canonical class of $R = R(X, \delta)$.*
If $a_t < m$ or $b_t < n$, then

$$\mathrm{cl}(\omega) = \mathrm{cl}(\omega_B R) + \textstyle\sum_{i=0}^{w} k_i \mathrm{cl}(I(x, \zeta_i)) + \sum_{i=0}^{z} l_i \mathrm{cl}(I(x, \eta_i)),$$

where $k_{i-1} - k_i = |\chi_{i-1}| - |\beta_i|$, for $i = 1, \dots, w$, $l_{i-1} - l_i = |\chi^*_{i-1}| - |\beta^*_i|$, for $i = 1, \dots, z$, and $l_z - k_w = |\chi^*_z| - |\chi_w|$ if $a_t < m$, $b_t < n$, $l_z - k_w = (|\chi^*_z| + |\beta_{w+1}|) - |\chi_w|$ if $a_t = m$, $b_t < n$, resp. $l_z - k_w = |\chi^*_z| - (|\beta^*_{z+1} + |\chi_w|)$ if $a_t < m$, $b_t = n$.
If $a_t = m$ and $b_t = n$, then

$$\mathrm{cl}(\omega) = \mathrm{cl}(\omega_B R) + \textstyle\sum_{i=0}^{w} k_i \mathrm{cl}(I(x, \zeta_i)) + \sum_{i=0}^{z} l_i \mathrm{cl}(I(x, \eta_i)) + k\mathrm{cl}(I(x, \gamma)),$$

where $k_{i-1} - k_i = |\chi_{i-1}| - |\beta_i|$, for $i = 1, \dots, w$, $l_{i-1} - l_i = |\chi^*_{i-1}| - |\beta^*_i|$, for $i = 1, \dots, z$, and $k - k_w = |\beta_{w+1}| - |\chi_w|$, $k - l_z = |\beta^*_{z+1}| - |\chi^*_z|$.

Note that the ring $R(X, \delta)$ is Gorenstein if and only if B is Gorenstein and all the differences in Theorem 2.3.1 vanish.

We single out the most important cases; see Bruns [13], [14], Bruns and Vetter [21], Eagon and Hochster [34], and Svanes [62] .

Theorem 2.3.2 *Let K be a field and X an $m \times n$ generic matrix over K. Then the classical determinantal ring $R_{t+1}(X)$ is a normal Cohen-Macaulay domain, its divisor class group is $\mathbb{Z}$, and it is Gorenstein if and only if $m = n$. Moreover, if $m \le n$ its canonical module is the ideal $P^{(n-m)}$, where P is the ideal of $R_{t+1}(X)$ generated by the t-minors of the first t rows of X.*

(S) To compute the canonical class of $R(X, \alpha)$ we shall use the isomorphism (2.3), induction on n, and the following lemma due to Bruns [21, Lemma (8.10)].

Lemma 2.3.3 *Let A be a normal Cohen-Macaulay domain, and I a prime ideal of height 1 in A such that A/I is again a normal Cohen-Macaulay domain. Let $Q_1, \dots, Q_u$ be prime ideals of height 1 in A and suppose that the class of I and the class of a canonical module ω_A have representations*

$$\mathrm{cl}(I) = \sum_{i=1}^{u} s_i \mathrm{cl}(Q_i) \quad \textit{and} \quad \mathrm{cl}(\omega_A) = \sum_{i=1}^{u} r_i \mathrm{cl}(Q_i).$$

Assume further that:
(i) $r_i - s_i \geq 0$ *for* $i = 1, \ldots, u$;
(ii) $\mathrm{Ann}(Q_i^{(r_i - s_i)}/Q_i^{r_i - s_i}) \not\subset Q_i + I$ *for* $i = 1, \ldots, u$;
(iii) The ideals $\overline{Q}_i = (Q_i + I)/I$ *are distinct prime ideals of height 1 in* A/I.

Then A/I *has a canonical module* $\omega_{A/I}$ *with*

$$\mathrm{cl}(\omega_{A/I}) = \sum_{i=1}^{u} (r_i - s_i)\mathrm{cl}(\overline{Q}_i).$$

We are now ready to compute the canonical class of $R(X, \alpha)$ and decide which one of these rings are Gorenstein.

Theorem 2.3.4 *Let* B *be a normal Cohen-Macaulay domain which admits a canonical module* ω_B, *and let* $\mathrm{cl}(\omega)$ *be a canonical class of* $R = R(X, \alpha)$.
If $\alpha_r < n$, *then*

$$\mathrm{cl}(\omega) = \mathrm{cl}(\omega_B R) + \textstyle\sum_{i=0}^{r} k_i \mathrm{cl}(I(x, \zeta_i)),$$

where $k_{i-1} - k_i = |\chi_{i-1}| - |\beta_i|$, *for* $i = 2, \ldots, r$ *and* $k_r \equiv |\chi_r| + 1 \bmod(2)$.
If $\alpha_r = n$, *then*

$$\mathrm{cl}(\omega) = \mathrm{cl}(\omega_B R) + \textstyle\sum_{i=0}^{r} k_i \mathrm{cl}(I(x, \zeta_i)),$$

where $k_{i-1} - k_i = |\chi_{i-1}| - |\beta_i|$, *for* $i = 2, \ldots, r-1$ *and* $k_{r-1} - 2k_r = |\chi_{r-1}| - |\beta_r| - 1$.

Proof. According to [21, Prop. (8.7)] we may assume that $B = K$ a field. Then the proof goes by induction on n. The case $n = 2$ is trivial. For $\alpha_t = n$ and $r = 1$, the ring $R(X, \alpha)$ is a polynomial ring, hence we may assume that $r > 1$ if $\alpha_t = n$. We may also assume $\alpha_1 = 1$; see the proof of Lemma 2.1.8.

The isomorphism (2.3) induces an isomorphism of divisor class groups

$$\psi^* : \mathrm{Cl}(R(X, \alpha)[x_{11}^{-1}]) \longrightarrow \mathrm{Cl}(R(\tilde{X}, \tilde{\alpha})).$$

The composition of the canonical epimorphism

$$\mathrm{Cl}(R(X, \alpha)) \longrightarrow \mathrm{Cl}(R(X, \alpha)[x_{11}^{-1}])$$

with ψ^* gives an epimorphism

$$g : \mathrm{Cl}(R(X, \alpha)) \longrightarrow \mathrm{Cl}(R(\tilde{X}, \tilde{\alpha})).$$

As long as the localization of a canonical module is also a canonical module, we get that g maps he canonical class to the canonical class.

For $\alpha_2 = 2$, we get by (2.4) that $g(\mathrm{cl}(I(x,\zeta_i))) = \mathrm{cl}(I(\tilde{x},\tilde{\zeta}_i))$ for all $i = 1,\ldots,r$. Now one applies the induction hypothesis.

For $\alpha_2 > 2$, we get by (2.5) that $g(\mathrm{cl}(I(x,\zeta_i))) = \mathrm{cl}(I(\tilde{x},\tilde{\zeta}_{i-1}))$ for all $i = 2,\ldots,r$ and $g(\mathrm{cl}(I(x,\zeta_1))) = 0$. By induction, it is enough to prove only the relation which involves k_1. We will distinguish three cases:

Case 1: $\alpha_t < n$ and $r = 1$. Then $\alpha = \{1\}$, and $R(X,\alpha) = R_2(X)$. But Goto [42] has proved that $R_2(X)$ is Gorenstein if and only if $n \equiv 0 \bmod(2)$. Therefore $k_1 \equiv n = |\chi_1| + 1 \bmod(2)$.

Case 2: $\alpha_t = n$ and $r = 2$. Set $\chi_1 = p-1$, with $n > p > 1$. Hence $\alpha = \{1, p{+}1, \ldots, n\}$. We will show that $k_1 - 2k_2 = p-1-(n-p)-1 = 2p-n-2$, i.e. $\mathrm{cl}(\omega) = (2p-n-2)\mathrm{cl}(I(x,\zeta_1))$.

Let X' be the submatrix of X of the first p rows. One denotes by S the ring $K[X']/I_2(X')$, where $I_2(X')$ is the ideal of $K[X']$ generated by all 2-minors of X'. One has that $I_2(X')K[X] = I(X,\alpha)$, and $S[X \setminus X'] = R(X,\alpha)$; see [22, Remark 2.5(c)]. One gets that S is a normal Cohen-Macaulay domain, and a canonical isomorphism $\mathrm{Cl}(S) \cong \mathrm{Cl}(R(X,\alpha))$ which maps the canonical class to the canonical class. The extension of the ideal $P = (x_{11},\ldots,x_{1n})$ of S to $R(X,\alpha)$ is $I(x,\zeta_1)$. Therefore it is sufficient to show that the canonical class of S is $\mathrm{cl}(\omega_S) = (2p-n-2)\mathrm{cl}(P)$. In order to prove it we will use Lemma 2.3.3.

Let $Q = (x_{ij}, 1 \le i \le j \le n, i \le p)$. It is immediate that Q is a prime ideal of height 1 in S, $\mathrm{cl}(P)$ and $\mathrm{cl}(Q)$ are generators of $\mathrm{Cl}(S)$, and the only relation is $2\mathrm{cl}(P) + \mathrm{cl}(Q) = 0$.

Let us first consider $p = 2$. We will argue by induction on n. For $n = 3$, $(2p-n-2)\mathrm{cl}(P) = -\mathrm{cl}(P) = \mathrm{cl}(P) + \mathrm{cl}(Q)$, therefore we have to show that the ideal $J = P \cap Q = (x_{11}, x_{12})$ is the canonical module of S. In this case $\dim(S) = 3$, and $S/J \cong K[X_{13}, X_{22}, X_{23}]/(X_{22}X_{13})$. Thus J is a maximal Cohen-Macaulay S-module. Therefore J is the canonical module of S if and only if its Cohen-Macaulay type $r(J)$ is 1. The sequence $\overline{x} = x_{11}, x_{23}, x_{13} - x_{22}$ is a system of parameters of S, hence $\overline{x}$ is a maximal regular S-sequence and J-sequence. Computing the Hilbert series of S and S/J, one shows that the Hilbert series of J is $2t + t^2/(1-t)^3$, hence the Hilbert series of $J/\overline{x}J$ is $2t + t^2$. The homogeneous component of degree 2 of $J/\overline{x}J$ is generated by $x_{11}x_{13} = x_{11}x_{22} = x_{12}^2 = x_{12}x_{13}$. It is clear that no 1-forms of $J/\overline{x}J$ annihilate the maximal ideal of S. So we get $r(J) = 1$.

Now suppose $n > 3$. We will apply 2.3.3 with respect to the ideal $I = (x_{1n}, x_{2n})$. It is obvious that $I \cong (x_{11}, x_{12}) = P \cap Q$, therefore $\mathrm{cl}(I) = \mathrm{cl}(P) + \mathrm{cl}(Q)$. We may write $\mathrm{cl}(\omega_S) = a\mathrm{cl}(P) + b\mathrm{cl}(Q)$, and we can assume that $a, b > 0$. Observe that the ideals P, Q are principal after inversion of x_{23}. Then a power of x_{23} annihilates $P^{(a-1)}/P^{a-1}$ and $Q^{(b-1)}/Q^{b-1}$. As $P + I$ and

$Q + I$ are prime ideals and do not contain the element x_{23}, the condition (ii) in 2.3.3 is fulfilled. By induction one gets $a - 1 - 2(b - 1) = 2 - (n - 1) - 2$, that is $a - 2b = 2 - n - 2$.

For $p > 2$, one argues again by induction using 2.3.3 with respect to the ideal $I = (x_{1p}, \dots, x_{pp}, x_{pp+1}, \dots, x_{pn})$. Note that $\mathrm{cl}(I) = \mathrm{cl}(P)$, and that P and Q are principal after inversion of x_{2n}. This concludes the Case 2.

Case 3: $\alpha_t < n$ and $r > 1$, or $\alpha_t = n$ and $r > 2$. We have to prove that $k_1 - k_2 = |\chi_1| - |\beta_2|$. Set $h = |\beta_2|$, and $C = \beta_3, \dots, \beta_r$. Then $\alpha = \{1, \alpha_2, \alpha_2 + 1, \dots, \alpha_2 + h - 1, C\}$. Set $\sigma = \{\alpha_2, \alpha_2 + 1, \dots, \alpha_2 + h, C\}$. Denote by y the residue class of the minor $[\sigma|\sigma]$ in $R(X, \alpha)$. By construction, $y \in I(x, \zeta_i)$ if and only if $\sigma \geq \zeta_i$ if and only if $i = 1$ or $i = 2$. In order to isolate k_1 and k_2 from the expression of $\mathrm{cl}(\omega)$, we will invert y. The class $k_1\mathrm{cl}(I(x, \zeta_1)) + k_2\mathrm{cl}(I(x, \zeta_2))$ is the canonical class of $R(X, \alpha)[y^{-1}]$. As in [21, Lemma (8.11)], we may identify the ring $R(X, \alpha)[y^{-1}]$ with a polynomial extension of a determinantal ring associated with the ideals of the 2-minors of a generic matrix of indeterminates. In this vein one considers the following sets:

$$\Psi_1 = \{[\sigma_i|\sigma_j] : 1 \leq i \leq j \leq n\} \cup \{[\sigma|\tilde{\sigma}] \in \Delta(X, \alpha) : \\ \tilde{\sigma} \text{ differs from } \sigma \text{ in exactly one index}\},$$

$$\Psi_2 = \{[\tilde{\sigma}|\sigma] \in \Delta(X, \alpha) : \tilde{\sigma} \text{ differs from } \sigma \text{ in exactly one index}\}.$$

Denote by $\tilde{\sigma}_{jk}$ the sequence $\{j, \alpha_2, \dots, (\alpha_2 + \widehat{h+1} - k), \dots, \alpha_2 + h, C\}$, and set $M_{jk} = [\tilde{\sigma}_{jk}|\sigma]$. Clearly $\Psi_2 = \{M_{jk} : 1 \leq j < \alpha_2, 1 \leq k \leq h + 1\}$. As in 2.1.3, expanding the minor $[i, \sigma|j, \sigma]$, one shows that $R(X, \alpha)[y^{-1}] = K[\Psi][y^{-1}]$. All we have to do is to determine the relations between the elements of Ψ. Actually, we claim that for all $1 \leq j, j_1 < \alpha_2$ and $1 \leq k \leq k_1 \leq h + 1$ we have

$$M_{jk}M_{j_1k_1} = [\tilde{\sigma}_{j\wedge j_1 k}|\tilde{\sigma}_{j\vee j_1 k_1}][\sigma|\sigma] = M_{jk_1}M_{j_1k},$$

where $j \wedge j_1 = \min\{j, j_1\}$ and $j \vee j_1 = \min\{j, j_1\}$.

First note that $M_{jk}M_{j_1k_1} = [\tilde{\sigma}_{jk}|\sigma][\sigma|\tilde{\sigma}_{j_1k_1}]$, since the matrix is symmetric. Then consider the "generic straightening relation" of $[\tilde{\sigma}_{jk}|\sigma][\sigma|\tilde{\sigma}_{j_1k_1}]$. Each standard monomials in generic standard representation of $[\tilde{\sigma}_{jk}|\sigma][\sigma|\tilde{\sigma}_{j_1k_1}]$ contains at most two factors. There are only two such standard monomials obtained from the indices of the minors under consideration, namely

$$[\tilde{\sigma}_{jk}|\tilde{\sigma}_{j_1k_1}][\sigma|\sigma] \text{ and } [j, \sigma|j_1, \sigma][\sigma \setminus \{\alpha_2 + h + 1 - k\}|\sigma \setminus \{\alpha_2 + h + 1 - k_1\}].$$

The first standard monomial appears in the representation with coefficient 1 (to see this specialize $j = k$ in the generic expression), and the second vanishes in $R(X, \alpha)$. Therefore

$$[\tilde{\sigma}_{jk}|\sigma][\sigma|\tilde{\sigma}_{j_1k_1}] = [\tilde{\sigma}_{jk}|\tilde{\sigma}_{j_1k_1}][\sigma|\sigma],$$

and it remains to show that $[\tilde{\sigma}_{jk}|\tilde{\sigma}_{j_1k_1}] = [\tilde{\sigma}_{j\wedge j-1k}|\tilde{\sigma}_{j\vee j_1k_1}]$. Because the matrix is symmetric, the equality is straightforward from the fact that the 2-minors of the first $(\alpha_2 - 1)$ rows are 0 in $R(X, \alpha)$.

Now we choose an $(\alpha_2-1)\times(h+1)$ matrix of indeterminates $T = (T_{ij})$, and an independent family $\{T_\psi : \psi \in \Psi_1\}$ of new indeterminates. Denote by a the determinant of the $t \times t$ symmetric matrix $(T_{[\sigma_1|\sigma_j]})$, and consider the following surjective ring homomorphism $w : K[T][T_\psi : \psi \in \Psi_1][a^{-1}] \longrightarrow R(X,\alpha)[a^{-1}]$, defined by $w(T_{ij}) = M_{ij}$, and $w(T_\psi) = \psi$. The ideal $I_2(T)$ is in the kernel of w, and a maps to y, so we get a surjective homomorphism

$$W : K[T]/I_2(T)[T_\psi : \psi \in \Psi_1][a^{-1}] \longrightarrow R(X,\alpha)[y^{-1}].$$

An element $\tilde{\sigma} \geq \sigma$ which differs from σ in exactly one index and has t entries is obtained from σ by replacing an index σ_i with an index $k > \sigma_i$ and $k \neq \sigma_j$ for $j > i$. We get $|\Psi_2| = t(t+1)/2 + \sum_{i=1}^t (n - \sigma_i - t + i) = t(n+1) - \sum_{i=1}^t \alpha_i - (\alpha_2 + h - 1) = \dim R - (\alpha_2 + h - 1)$. On the other hand, $\dim K[T]/I_2(T)[T_\psi : \psi \in \Psi_1][a^{-1}] = |\Psi_2| + (\alpha_2 + h - 1)$, therefore W is an isomorphism.

Furthermore the prime ideal P of $K[T]/I_2(T)$ generated by the elements of the first row of T extends to $I(x,\zeta_1)R(X,\alpha)[y^{-1}]$, and the prime ideal Q generated by the elements of the first column extends to $I(x,\zeta_2)R(X,\alpha)[y^{-1}]$. By construction we have

$$w(T_{1k}) \in I(x,\zeta_1)R(X,\alpha)[y^{-1}] \text{ and } w(T_{j1}) \in I(x,\zeta_2)R(X,\alpha)[y^{-1}].$$

Then the extension of P is contained in $I(x,\zeta_1)R(X,\alpha)[y^{-1}]$, and both are prime ideals of height 1, so that they must coincide.

Therefore we get an isomorphism between the divisor class groups

$$\mathrm{Cl}(K[T]/I_2(T)) \longrightarrow \mathrm{Cl}(R(X,\alpha)[y^{-1}]),$$

and we deduce that $k_1\mathrm{cl}(P) + k_2\mathrm{cl}(Q)$ is the canonical class of $K[T]/I_2(T)$. From [21, Theorem (8.8)] one deduces that $k_1 - k_2 = (\alpha_2 - 1) - (h+1) = (\alpha_2 - 2) - h = |\chi_1| - |\beta_2|$. □

Corollary 2.3.5 *Let B be a Noetherian ring. Then the ring $R = R(X,\alpha)$ is Gorenstein if and only if B is Gorenstein and*
$|\chi_{i-1}| - |\beta_i| = 0$, for all $i = 2,\dots,r$, and $k_r + 1 \equiv 0 \bmod(2)$ if $\alpha_t < n$, or
$|\chi_{i-1}| - |\beta_i| = 0$, for $i = 2,\dots,r-1$ and $k_{r-1} - |\beta_r| - 1 = 0$.

We single out the most important cases; see Kutz [53], and Goto [41], [42].

Theorem 2.3.6 *Let K be a field and X an $n \times n$ generic symmetric matrix over K. Let $1 \leq t < n$. Then the classical ring $S_{t+1}(X)$ of symmetric minors is a normal Cohen-Macaulay domain, its divisor class group is $\mathbb{Z}_2$, and it is Gorenstein if and only if $t+1 \equiv n \bmod(2)$. Moreover, if $t+1 \not\equiv n \bmod(2)$, its canonical module is the ideal P generated by the t-minors of the first t rows of X, and its Cohen-Macaulay type is $\binom{n}{t}$.*

Proof. One observes that a minimal system of generators of P is given by the t-minors of the first t rows, since are all doset minors. The rest follows easily from 2.2.3 and 2.3.4. □

(**A**) We are going to compute the canonical class of $R(X,\pi)$ and decide which one of these rings are Gorenstein.

Theorem 2.3.7 *Let B be a normal Cohen-Macaulay domain which admits a canonical module ω_B, and let $\mathrm{cl}(\omega)$ be a canonical class of $R = R(X,\pi)$.*
If $a_{2t} < n$ or $a_{2t} = n$ and $a_{2t-1} < n-1$, then

$$\mathrm{cl}(\omega) = \mathrm{cl}(\omega_B R) + \textstyle\sum_{i=0}^{u} k_i \mathrm{cl}(I(x,\zeta_i)),$$

where $k_{i-1} - k_i = |\chi_{i-1}| - |\beta_i|$, for all $i = 1,\dots,u$.
If $a_{2t} = n$ and $a_{2t-1} = n-1$, then

$$\mathrm{cl}(\omega) = \mathrm{cl}(\omega_B R) + \textstyle\sum_{i=0}^{s-1} k_i \mathrm{cl}(I(x,\zeta_i)) + k\mathrm{cl}(I(x,\zeta)),$$

where $k_{i-1} - k_i = |\chi_{i-1}| - |\beta_i|$, for all $i = 1,\dots,s-1$ and $k_{s-1} - k = |\chi_{s-1}| - |\beta_s| + 1$.

Proof. According to [21, Prop. (8.7)] we may assume that $B = K$ a field.
In the first case, remember that the upper neighbours of π are

$$\zeta_i = [\beta_0,\dots,\beta_{i-1},(a_{k_i+1},\dots,a_{k_{i+1}-1}),a_{k_{i+1}}+1,\beta_{i+1},\dots,\beta_s],$$

$i = 0,\dots,u$. (Of course, $u = s$ if $a_{2t} < n$ and $u = s-1$ if $a_{2t} = n$ and $a_{2t-1} < n$.) Since $\sum_{i=0}^{u} \mathrm{cl}(I(x,\zeta_i)) = 0$, the differences $k_{i-1} - k_i, i = 0,\dots,u$ determine $\mathrm{cl}(\omega)$ uniquely. In order to isolate k_{i-1} and k_i in the representation of $\mathrm{cl}(\omega)$, let us consider the elements σ_i in $\Pi(X,\pi)$

$$\sigma_i = [\beta_0,\dots,\beta_{i-2},\ (a_{k_{i-1}+1},\ \dots,\ a_{k_i-1}),\ (a_{k_i+1},\ \dots,\ a_{k_{i+1}},\ a_{k_{i+1}}+1),\ \dots,\ \beta_{i+1},\ \dots,\ \beta_s]$$

for all $i = 1,\dots,u$. Let $S = R[\sigma_i^{-1}]$. Because $\zeta_j \le \sigma_i$ if and only if $j = i-1$ or $j = i$, the ideals $I(x,\zeta_{i-1})$ and $I(x,\zeta_i)$ are the only minimal ideals of π which survive in S. Then

$$\mathrm{cl}(\omega_S) = k_{i-1}\mathrm{cl}(I(x,\zeta_{i-1})S) + k_i\mathrm{cl}(I(x,\zeta_i)S).$$

Following the ideas of Bruns and Vetter (see [21, pg.101-102]), we get that the ring S is isomorphic to a polynomial extension of a determinantal ring associated to the ideal of the 2-minors of a generic matrix of indeterminates. In order to do this we consider that $\sigma_i = [\alpha_1,\dots,\alpha_{2t}]$ and let $\Psi = \{[\alpha_i\alpha_j] : 1 \le i < j \le 2t\} \cup \{\tilde{\delta} \in \Pi(X,\pi) : \tilde{\delta}$ differs from σ_i in exactly one index$\}$. Almost everything goes as in [21, Lemma (8.11)] (instead of Plücker relations we must use Lemma 6.1 from [28]), and finally we get that $k_{i-1}-k_i = |\chi_{i-1}|-|\beta_i|$. One should observe that the similarity with Lemma (8.11) from [21] is preserved in this case just because we deal only with $2t$-pfaffians. This is not the case in the second part of the proof.

As we have seen before, in the second case, when $a_{2t}=n$ and $a_{2t-1}=n-1$, once with ζ_i, $i=0,\dots,s-1$ appears another upper neighbour of π, namely $\zeta=[a_1,\dots,a_{2t-2}]$. In order to isolate k_{s-1} and k in the representation of $\mathrm{cl}(\omega)$, we should consider now a $(2t-2)$-pfaffian σ in $\Pi(X,\pi)$

$$\sigma=[a_1,\dots,a_{k_s-1},a_{k_s+1},\dots,a_{2t-1}].$$

Of course, $\zeta_j\le\sigma$ if and only if $j=s-1$ and moreover $\zeta\le\sigma$. So the ideals $I(x,\zeta_{s-1})$ and $I(x,\zeta)$ are the only minimal prime ideals of π which survive in $S=R[\sigma^{-1}]$. To determine the structure of the ring S we shall introduce the set $\Psi=\{[\alpha_i\alpha_j]:1\le i<j\le 2t-2\}\cup\{\tilde\delta\in\Pi(X,\pi):\tilde\delta$ differs from σ in exactly one index$\}$, where $\sigma=[\alpha_1,\dots,\alpha_{2t-2}]$, and subdivide it into two subsets $\Psi_1=\{\tilde\delta\in\Psi:\tilde\delta\ge\sigma\}$ and $\Psi_2=\{\tilde\delta\in\Psi:\tilde\delta\not\ge\sigma\}$. We get

$$\Psi_2=\{[a_1,\dots,a_{k_s-1},a_{k_s}+j-1,a_{k_s+2},\dots,a_{2t-1}]:1\le j\le a_{k_s+1}-a_{k_s}\},$$

so that $|\Psi_2|=a_{k_s+1}-a_{k_s}$. On the other hand, Ψ_1 is quite similar to the set defined in 2.1.11, therefore $|\Psi_1|=\dim R(X,\sigma)-\dim B$. Let $p=a_{k_s+1}-a_{k_s}$ and $q=2t-k_s$. We choose a $p\times q$ matrix T of indeterminates, and an independent family $\{T_\psi:\psi\in\Psi_1\}$ of indeterminates over B. Analogous to [21, Lemma (8.11)] we claim that the substitution $T_\psi\longrightarrow\psi$, $\psi\in\Psi_1$,

$$T_{j1}\longrightarrow[a_1,\dots,a_{k_s-1},a_{k_s}+j-1,a_{k_s+1},\dots,a_{2t}],$$

and

$$T_{jk}\longrightarrow[a_1,\dots,a_{k_s-1},a_{k_s}+j-1,a_{k_s+1},\dots,\hat a_{k_s+k-1},\dots,a_{2t-1}],$$

for $k=2,\dots,2t-k_s$ induces an isomorphism

$$(B[T]/I_2(T))[T_\psi:\psi\in\Psi_1][f^{-1}]\longrightarrow R[\sigma^{-1}],$$

where $f=\mathrm{pf}((X_{\alpha_i\alpha_j})_{1\le i\le j\le 2t-2})$.

We should observe that the substitution maps f to σ, so we get a homomorphism

$$\varphi:B[T][T_\psi:\psi\in\Psi_1][f^{-1}]\longrightarrow R[\sigma^{-1}],$$

and its image contains $B[\Psi][\sigma^{-1}]$. In the same way as in 2.1.11 we get that every element $[uv]\in\Pi(X,\pi)$ belongs to $B[\Psi][\sigma^{-1}]$, except the case when $u=a_{k_s}+j-1$, $1\le j\le p$ and $v=n$. In this case the element $[u,v,\alpha_1,\dots,\alpha_{2t-2}]$ is the image of T_{j1} by φ, and it expands in $R(X,\pi)$ with respect to the first two rows in a sum which contains the term $[uv]\sigma$ and another terms which belong to the image of φ, so $[uv]$ is in the image of φ, too.

Now we prove that $I_2(T)$ is in the kernel of φ. This means that the image of the product of indeterminates T_{jk} and T_{lm} coincides with the image of the product of T_{jm} and T_{lk}. There are two cases to consider.

Case 1: $k = 1$ or $m = 1$. Let us consider that $k = 1$ and $m > 1$. Then the product in $R(X, \pi)$ of the pfaffians

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + j - 1, a_{k_s+1}, \dots, a_{2t}]$$

and

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + l - 1, a_{k_s+1}, \dots, \hat{a}_{k_s+m-1}, \dots, a_{2t-1}]$$

can be expanded using [28, Lemma 6.1]. We get that this product coincides with the product of pfaffians

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + l - 1, a_{k_s+1}, \dots, a_{2t}]$$

and

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + j - 1, a_{k_s+1}, \dots, \hat{a}_{k_s+m-1}, \dots, a_{2t-1}],$$

because the other two terms of the sum which possibly appear should contain the pfaffians

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + j - 1, a_{k_s} + l - 1, a_{k_s+1}, \dots, \hat{a}_{k_s+m-1}, \dots, a_{2t}]$$

and

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + j - 1, a_{k_s} + l - 1, a_{k_s+1}, \dots, a_{2t-1}]$$

respectively, which drop out in $R(X, \pi)$.

Case 2: $k > 1$ and $m > 1$. In this case we have to consider the product of two $(2t-2)$-pfaffians, namely

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + j - 1, a_{k_s+1}, \dots, \hat{a}_{k_s+k-1}, \dots, a_{2t-1}]$$

and

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + l - 1, a_{k_s+1}, \dots, \hat{a}_{k_s+m-1}, \dots, a_{2t-1}].$$

This product is expanded again using [28, Lemma 6.1], and one observes that all the terms we can get contain a pfaffian which drops out in $R(X, \pi)$ (this pfaffian contains the indices $a_{k_s} + j - 1$ and $a_{ks} + l - 1$), except the product which involves the pfaffians

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + j - 1, a_{k_s+1}, \dots, \hat{a}_{k_s+m-1}, \dots, a_{2t-1}]$$

and

$$[a_1, \dots, a_{k_s-1}, a_{k_s} + l - 1, a_{k_s+1}, \dots, \hat{a}_{k_s+k-1}, \dots, a_{2t-1}].$$

Because the ideal $I_2(T)$ is in the kernel of φ, we get a surjective homomorphism

$$(B[T]/I_2(T))[T_\psi : \psi \in \Psi_1][f^{-1}] \longrightarrow R[\sigma^{-1}].$$

Note that $\dim(B[T]/I_2(T))[T_\psi : \psi \in \Psi_1][f^{-1}] = \dim B + (p+q-1) + |\Psi_1|$. But $|\Psi_1| = \dim R(X,\sigma) - \dim B$ (see 2.1.11), so we have $\dim(B[T]/I_2(T))[T_\psi : \psi \in \Psi_1][f^{-1}] = (p+q-1) + \dim R(X,\sigma) = (p+q-1) + \dim B + 2n(t-1) - \sum_{i=1}^{2t-2} \alpha_i = (p+q-1) + \dim B + 2nt - \sum_{i=1}^{2t} a_i + (a_{k_s} - n) = (p+q-1) + (a_{k_s} - n) + \dim R = \dim R = \dim R[\sigma^{-1}]$. Therefore the surjective homomorphism above is an isomorphism.

Furthermore the prime ideal P generated in $B[T]/I_2(T)$ by the elements of the first row of T extends to $I(x, \zeta_{s-1})S$, and the prime ideal Q generated by the elements of the first column extends to $I(x,\zeta)S$. By construction $\varphi(T_{j1}) \in I(x,\zeta)S$, and $\varphi(T_{1k}) \in I(x,\zeta_{s-1})S$. Then the extension of P is contained in $I(x,\zeta_{s-1})S$, but since both are prime ideals of height 1 they must coincide, and the same argument works as well for Q.

We get an isomorphism between the divisor class groups

$$\mathrm{Cl}(B[T]/I_2(T)) \longrightarrow \mathrm{Cl}(S),$$

therefore we can deduce that $k_{s-1}\mathrm{cl}(P) + k\mathrm{cl}(Q)$ is the canonical class of $B[T]/I_2(T)$. By [21, Theorem (8.8)], we conclude that $k_{s-1} - k = p - q = (p-1) - q + 1 = |\chi_{s-1}| - |\beta_s| + 1$. □

Corollary 2.3.8 *Let B be an (arbitrary) Noetherian ring. Then $R(X,\pi)$ is Gorenstein if and only if B is Gorenstein and*
$|\chi_{i-1}| - |\beta_i| = 0$ for all $i = 1, \dots, u$ if $a_{2t} < n$ or $a_{2t} = n$ and $a_{2t-1} < n-1$, or $|\chi_{i-1}| - |\beta_i| = 0$ for all $i = 1, \dots, s-1$ and $|\chi_{s-1}| - |\beta_s| + 1 = 0$ if $a_{2t} = n$ and $a_{2t-1} = n-1$.

We single out the most important cases; see Avramov [3], and Kleppe and Laksov [50].

Theorem 2.3.9 *Let B be a field and X an $n \times n$ generic alternating matrix over B. Then the classical ring of pfaffians $P_{t+1}(X)$ is a normal Cohen-Macaulay domain. Moreover, it is factorial and therefore Gorenstein.*

Proof. As we have seen in Section 2, $\mathrm{Cl}(P_{t+1}(X)) = 0$. Therefore P_{t+1} is factorial, and by Murthy's theorem (see [37, §12]) it must be Gorenstein. □

Chapter 3

Gröbner bases and determinantal ideals

3.1 Bitableaux, combinatorial alghoritms, and the Knuth-Robinson-Schensted correspondence

The Knuth-Robinson-Schensted correspondence, KRS for short, is a one-to-one correspondence between standard (Young) bitableaux and two-line arrays of positive integers of a certain type. It is constructed by Knuth [51] relying on a combinatorial algorithm of Schensted [58].

We first consider the (bi)tableaux as purely combinatorial objects, that is, we call a *(Young) tableau* an array of positive integers $A = (a_{ij})$, $1 \le i \le u$, $1 \le j \le r_i$ with $r_1 \ge \cdots \ge r_u$. Such a tableau is said to be *standard* if it has the numbers in each row are in strictly increasing order from left to right, and the numbers in each column are in non-decreasing order from top to bottom (that is $a_{ij} < a_{ij+1}$ for all $i = 1, \ldots, u$, $j = 1, \ldots, r_i - 1$ and $a_{ij} \le a_{i+1j}$ for all $i = 1, \ldots, u-1$, $j = 1, \ldots, r_{i+1}$). The *shape* of the tableau A is the sequence $\lambda = (r_1, \ldots, r_u)$, and the *length* is r_1. (We will often call shape a non-increasing sequence of integers without reference to a tableau.) A *standard bitableau* is an ordered pair $\Sigma = (A, B)$ of standard tableaux of the same shape. We define $\min(\Sigma)$ as being the bitableau formed by the first rows of A and B.

We shall describe Schensted's algorithm DELETE (for deleting a place from a standard tableau) and its inverse INSERT (for inserting an integer in a standard tableau).

Definition 3.1.1 The inputs of DELETE are a standard tableau $A = (a_{ij})$ of shape $(r_1, \ldots, r_u)$ and an index i, $i = 1, \ldots, u$, such $r_i > r_{i+1}$. The outputs are a standard tableau B of shape $(r_1, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_u)$ and an element

$a \in A$. The element a is determined successively as follows. Set $a_i = a_{ir_i}$, and for $j = i-1, \ldots, 1$ let a_j be the largest element of the jth row of A less than or equal to a_{j+1}. Then we define B as the standard tableau obtained from A replacing a_j with a_{j+1} in the jth row for $j = i, \ldots, 2$. Then set $a = a_1$ and remove the element a_i in the ith row.

Let us consider the following example.

Example 3.1.2 Let

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & \\ 3 & 5 & \\ 4 & & \end{pmatrix}$$

be a standard tableau of shape (3,2,2,1), and set $i = 4$. Then $a_4 = 4$, $a_3 = 3$, $a_2 = 3$, $a = a_1 = 2$, and

$$B = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 3 & \\ 4 & 5 & \end{pmatrix}.$$

Definition 3.1.3 The inputs of INSERT are a standard tableau $B = (b_{ij})$ of shape $(r_1, \ldots, r_u)$ and an integer a. The outputs are a standard tableau A whose shape is $(r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_u)$ and an integer i. We define successively a sequence of integers. Set $a_1 = a$. Then suppose $a_1, \ldots, a_j$ has already been defined. If $a_j > b_{jr_j}$ or $j = u+1$, then the sequence terminates. If $a_j \leq b_{jr_j}$, let a_{j+1} be the minimum of the elements of the jth row of B greater than or equal to a_j. Thus we have got a sequence $a_1, \ldots, a_i$ and $a_i > b_{ir_i}$ or $i = u + 1$. Then we define A to be the standard tableau obtained from B by replacing a_{j+1} with a_j for $j = 1, \ldots, i-1$, and by adding a_i in the place $(i, r_i + 1)$.

Example 3.1.4 Let

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & \\ 3 & 6 & 8 & \\ 5 & 6 & & \\ 8 & & & \end{pmatrix}$$

be a standard tableau of shape (4,3,3,2), and set $a = 3$. Then $a_1 = 3$, $a_2 = 5$, $a_3 = 6$, $a_4 = 8$, $i = 4$ and

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 8 \\ 3 & 5 & 6 & \\ 3 & 6 & & \\ 8 & & & \end{pmatrix}.$$

The algorithms DELETE and INSERT are clearly inverse one to each other. If DELETE applied to input (A, i) gives the output (B, a), then INSERT applied to (B, a) gives output (A, i).

As we claimed the Knuth-Robinson-Schensted correspondence is a one-to-one correspondence between the set of the standard (Young) bitableaux and the set of the two-line arrays of positive integers of a certain type. The two-line array KRS (Σ) is constructed from the non-empty standard tableau Σ of shape $(r_1, \ldots, r_u)$ by iteration.

Definition 3.1.5 Set $r = r_1 + \cdots + r_u$ and define the two-line array

$$\mathrm{KRS}(\Sigma) = \begin{pmatrix} u_1 & & u_r \\ v_1 & & v_r \end{pmatrix}$$

which satisfies the conditions $u_1 \leq \cdots \leq u_r$ and $v_i \geq v_{i+1}$ if $u_i = u_{i+1}$ by using a procedure that determines the elements u_i and v_i recursively via a sequence of standard bitableaux $\Sigma_i = (A_i, B_i)$ as follows. Starting with $\Sigma_r = \Sigma$, for $i = r, \ldots, 1$:

- Let u_i be the maximum of the elements of A_i. Suppose that s is the largest index such that u_i can be found on the sth row of A_i. Then define A_{i-1} as being the standard tableau obtained from A_i by removing the element in the sth row.

- Apply DELETE to the pair (B_i, s). The outputs will be v_i and B_{i-1}.

- Set $\Sigma_{i-1} = (A_{i-1}, B_{i-1})$

Let us give one more example.

Example 3.1.6 Let $\Sigma = (A, B)$, where

$$A = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & & \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 4 & 5 & & \end{pmatrix}.$$

Then $u_6 = 6$, $v_6 = 5$,

$$\Sigma_5 = (A_5, B_5) = \left(\begin{array}{cccc|cccc} 1 & 3 & 4 & 5 & 1 & 2 & 5 & 6 \\ 2 & & & & 4 & & & \end{array} \right),$$

$u_5 = 5$, $v_5 = 6$,

$$\Sigma_4 = (A_4, B_4) = \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 2 & 5 \\ 2 & & & 4 & & \end{array} \right),$$

$u_4 = 4$, $v_4 = 5$,

$$\Sigma_3 = (A_3, B_3) = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 2 \\ 2 & & 4 & \end{array} \right),$$

$u_3 = 3$, $v_3 = 2$,

$$\Sigma_2 = (A_2, B_2) = \left(\begin{array}{c|c} 1 & 1 \\ 2 & 4 \end{array} \right),$$

$u_2 = 2$, $v_2 = 4$,

$$\Sigma_1 = \left(1 \,|\, 4 \right),$$

$u_1 = 1$, $v_1 = 4$.

Consequently, we get

$$\mathrm{KRS}(\Sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 2 & 5 & 6 & 3 \end{pmatrix}.$$

To get the inverse of KRS one just applies the algorithm INSERT to the bottom line of the array to build the right tableau: at step i it inserts v_i in the tableau obtained after the step $i-1$. In the same time the left tableau is built by placing the element u_i in the position which is added to the right tableau by the ith step of INSERT.

3.2 Schensted and Greene's theorems

The main result in Schensted [58] deals with the determination of the length of the longest increasing (decreasing) subsequence of a given sequence of integers. Let $v = (v_1, \ldots, v_r)$ be a sequence of integers and let INSERT(v) be the standard tableau determined by the iterated insertions of the v_i. A subsequence $v_{i_1}, \ldots, v_{i_k}$ of v with $i_1 < \cdots < i_k$ is called *increasing* (resp. *decreasing*) if $v_{i_1} < \cdots < v_{i_k}$ (resp. $v_{i_1} > \cdots > v_{i_k}$). The *length* of a sequence is the number of its elements.

There is a close relationship between the shape of INSERT(v) and the sequence v given by the following theorem of Schensted [58].

Theorem 3.2.1 *The length of the longest increasing (resp. decreasing) subsequence of v is the length of the first row (resp. column) in the tableau* INSERT(v).

However, later on it will be useful to have an interpretation for the rest of the shape of INSERT(v). We may ask whether the length of the ith row (column) for $i > 1$ have a similar meaning. The answer is yes, but we cannot interpret them individually. Actually, some of their partial sums give information on the decompositions of the sequence v into increasing (decreasing) subsequences. This connection is the content of Greene's extension of Schensted theorem; see Greene [43]. At this moment an observation of Schensted [58] is in order: we can replace any sequence by a sequence of distinct integers such as the increasing (decreasing) subsequences of these two sequences will be in one-to-one correspondence. Moreover, if we perform the algorithm INSERT on both sequences we get two standard tableaux with the same shape.

Let us now enter the details and explain the theorem of Greene. For a sequence $v = (v_1, \ldots, v_r)$ of integers and $k \le r$, denote by $d_k(v)$ the length of the longest subsequence of v which has no increasing subsequences of length $k+1$. It is rather easy to see that any such sequence is obtained by taking the union of k decreasing subsequences. Similarly, we define $a_k(v)$ to be the length of the longest subsequence of v which has no decreasing subsequences of length $k+1$. Note that $a_1(v)$, respectively $d_1(v)$ represent the length of the longest increasing, respectively decreasing subsequence of v. If $\lambda = (r_1, \ldots, r_u)$ is the shape of a tableau A we define $\lambda^* = (r_1^*, \ldots, r_{r_1}^*)$ to be the *dual* shape, that is r_i^* is the length of the ith column of A, equivalently r_i^* is the number of rows of A of length at least i. We are now ready to state Greene's theorem.

Theorem 3.2.2 *For every sequence of integers v and every $k \ge 1$ we have*

$$a_k(v) = r_1 + \cdots + r_k \text{ and } d_k(v) = r_1^* + \cdots + r_k^*,$$

where $\lambda = (r_1, \ldots, r_u)$ is the shape of INSERT(v) *and $\lambda^* = (r_1^*, \ldots, r_{r_1}^*)$ is the dual shape of λ.*

3.3 KRS and generic matrices

(G) Next we describe a correspondence between standard bitableaux and monomials of the set of indeterminates $X = \{X_{ij} : i, j = 1, 2, \ldots\}$. Note that every monomial $\prod X_{ij}^{k_{ij}}$ can be uniquely rewritten in the form $X_{u_1v_1} \cdots X_{u_rv_r}$ with $u_1 \le \cdots \le u_r$ and $v_i \ge v_{i+1}$ if $u_i = u_{i+1}$. Therefore we have a one-to-one correspondence between the set of monomials in X and the set of two-line arrays of positive integers with the above mentioned properties.

We shall identify a product of minors of a generic matrix $X = (X_{ij})$ with a bitableau, that is, whenever $\mu = \delta_1 \cdots \delta_u$ with $\delta_i = [a_{i1}, \ldots, a_{ir_i} | b_{i1}, \ldots, b_{ir_i}]$ an r_i-minor, $r_1 \ge \cdots \ge r_u$, we consider μ as being the bitableau $\Sigma = (A, B)$, where $A = (a_{ij})$ and $B = (b_{ij})$ are tableaux. It is obvious that the standard monomials correspond to the standard bitableaux.

If we restrict our attention to standard bitableaux and monomials whose entries, respectively indeterminates come from an $m \times n$ generic matrix X, we get the following

Theorem 3.3.1 KRS *is a one-to-one correspondence between the set of standard bitableaux on the set* $\{1,\dots,m\}\times\{1,\dots,n\}$ *and the monomials in* $K[X]$.

Note that Theorem 3.3.1 implies that KRS is a bijection between two distinct bases of the K-vector space $K[X]$, namely the *standard* base and the monomial base. Therefore we can extend KRS to a K-linear automorphism of $K[X]$, and we will often take this point of view in the sequel. The automorphism KRS preserves the total degree, no column or row index disappears, but however it is not a K-algebra isomorphism: it acts as the identity on indeterminates but it is not the identity map.

Remark 3.3.2 (a) The definition of standard tableaux and the algorithms DELETE and INSERT correspond to the definition of dual tableau and to the algorithms DELETE* and INSERT* in the Knuth's paper [51]. In particular, Knuth treats the KRS correspondence for *column standard* bitableau with increasing columns and non-decreasing rows. (This will be our point of view in part (A)). The above version is the "dual" version, in terms of Knuth; see Knuth [51, Section 5].

(b) The KRS correspondence is also extensively treated by Fulton [38], Sagan [57], and Stanley [60]. These books are highly recommended for the interested reader in the applications of KRS in combinatorics and the group representations theory.

We end with some important properties of KRS.

Lemma 3.3.3 KRS *commutes with the transposition of a matrix, that is, whether* $\varepsilon : K[X] \longrightarrow K[X]$ *denote the* K*-algebra isomorphism induced by the substitution* $X_{ij} \longmapsto X_{ji}$*, then* $\mathrm{KRS}(\varepsilon(f)) = \varepsilon(\mathrm{KRS}(f))$ *for all* $f \in K[X]$.

Proof. It is sufficient to prove it for f a standard bitableau. The argument is essentially the same as in Knuth [51, Theorem 3]; see also Herzog and Trung [45, Lemma 1.1]. □

In the application of KRS to find Gröbner bases of determinantal ideals the next result is a key step.

Lemma 3.3.4 *Let* Σ *be a standard bitableau, and* $[a_{11},\dots,a_{1r_1}|b_{11},\dots,b_{1r_1}]$ *its first row. If* $\mathrm{KRS}(\Sigma) = X_{u_1v_1}\cdots X_{u_rv_r}$*, then for every* $s = 1,\dots,r_1$ *there exists a factor* $X_{u_{\alpha_1}v_{\alpha_1}}\cdots X_{u_{\alpha_s}v_{\alpha_s}}$ *of* $\mathrm{KRS}(\Sigma)$ *such that* $u_{\alpha_1} < \cdots < u_{\alpha_s}$*,* $v_{\alpha_1} < \cdots < v_{\alpha_s}$ *and* $u_{\alpha_s} = a_{1s}$ *(respectively* $v_{\alpha_s} = b_{1s}$*).*

Proof. Let $\Sigma_i = (A_i, B_i)$, $i = 1,\dots,r$, be the standard bitableaux occurring in the above described procedure of computing $\mathrm{KRS}(\Sigma)$. It is not difficult to see that

$$\mathrm{KRS}(\Sigma_i) = \begin{pmatrix} u_1 & & u_i \\ v_1 & & v_i \end{pmatrix}.$$

Furthermore, there is a step of the procedure when $u_i = a_{1s}$, and A_{i-1} is obtained from A_i by removing the element a_{1s}. So a_{1s} is the maximum of the elements of A_i, hence the first row of A_i must end by a_{1s}. It follows that the length of the first row of A_i, resp. A_{i-1} is s, resp. $s-1$. According to Theorem 3.2.1, the length of the longest increasing subsequence of $v_1, \ldots, v_i$ is s, and that of $v_1, \ldots, v_{i-1}$ is $s-1$. Let $v_{\alpha_1} < \cdots < v_{\alpha_s}$ be an increasing subsequence of $v_1, \ldots, v_i$. It follows that $\alpha_s = i$, hence $u_{\alpha_s} = u_i = a_{1s}$. As $v_{\alpha_1} < \cdots < v_{\alpha_s}$, we also get $u_{\alpha_1} < \cdots < u_{\alpha_s}$ since KRS (Σ) satisfies the conditions of Definition 3.1.5, and the proof is done. □

(S) In this part we define a modification of the Knuth-Robinson-Schensted correspondence that suits to the generic symmetric case, but the definition of (standard) tableaux and the definition of the algorithms DELETE and INSERT are the same as in part (G).

Definition 3.3.5 A (standard) tableau A of shape $r_1, r_1, \ldots, r_u, r_u$ is called (standard) *tableau of double shape*, or (standard) *d-tableau* for short.

Let us now define a KRS type correspondence for standard d-tableaux. Let A be a standard d-tableau of shape $r_1, r_1, \ldots, r_u, r_u$ and set $r = r_1 + \cdots + r_u$. We define recursively a sequence of standard d-tableaux $A_r, \ldots, A_1, A_0$, where $A_0 = \emptyset$, and two sequences of integers $u_r, \ldots, u_1$ and $v_r, \ldots, v_1$. First set $A_r = A$. Then, for $i = r, \ldots, 1$, let u_i be the maximum of the elements of A_i, and p_i the largest index with the property that u_i can be found in the p_ith row of A_i. Since A_i is a standard d-tableau the integer p_i is even, say $p_i = 2s_i$. Then

(1) Apply DELETE to input (A_i, p_i) and let (A'_i, v_i) be the output.

(2) Cancel u_i from the tableau A'_i in the $(p_i - 1)$th row to get A_i.

Remark that whether the shape of A_i is $r_1, r_1, \ldots, r_q, r_q$, then the shape of A'_i is $r_1, r_1, \ldots, r_{s_i}, r_{s_i} - 1, \ldots, r_q, r_q$ and u_i is in the $(p_i - 1)$th row of A'_i. Finally A_{i-1} is a standard d-tableau of shape $r_1, r_1, \ldots, r_{s_i} - 1, r_{s_i} - 1, \ldots, r_q, r_q$.

Then we define

$$\phi(A) = \begin{pmatrix} u_1 & & u_r \\ v_1 & & v_r \end{pmatrix}.$$

Note that the maximum of the elements of the $(p_i - 1)$th row of A_i which are less than or equal to u_i is the last element of this row. Therefore we can replace the steps (1) and (2) by the steps

(1*) Cancel u_i from the tableau A_i in the p_ith row to get the standard tableau A^*_i.

(2*) Apply DELETE to input $(A^*_i, p_i - 1)$ and let (A_{i-1}, v_i) be the output.

Remark again that whether the shape of A_i is $r_1, r_1, \dots, r_q, r_q$, then the shape of A_i^* is $r_1, r_1, \dots, r_{s_i}, r_{s_i} - 1, \dots, r_q, r_q$.

It is obvious that $v_i \le u_i$ and that $u_1, \dots, u_r$. In order to show that the two-line array $\phi(A)$ satisfies the condition $v_i \ge v_{i+1}$ if $u_i = u_{i+1}$, we observe that $u_i = u_{i+1}$ implies $p_< p_{i+1}$. By using first (2*) and then (1), we get DELETE$(A_{i+1}^*, p_{i+1} - 1)$=(A_i, v_{i+1}) and DELETE(A_i, p_i)=(A_i', v_i). Note that $p_i < p_{i+1} - 1$ since p_i and p_{i+1} are even integers. Therefore we get $v_i \ge v_{i+1}$ as a consequence of the following theorem of Knuth [51, Theorem 1*].

Theorem 3.3.6 *Let B be a standard tableau and i an integer such that we can apply* DELETE(B, i). *Then*
(a) *If* DELETE$(B, i) = (B_1, a)$ *and* DELETE$(B_1, j) = (B_2, b)$, *then $i > j$ if and only if $a \le b$.*
(b) *If* INSERT$(B_2, b) = (B_1, j)$ *and* INSERT$(B_1, a) = (B, i)$, *then $i > j$ if and only if $a \le b$.*

Consequently ϕ defines a map between the set of standard d-tableaux and the set of the two-line arrays of positive integers

$$\begin{pmatrix} u_1 & \dots & u_r \\ v_1 & \dots & v_r \end{pmatrix}$$

which satisfies the conditions $u_1 \le \dots \le u_r$, $v_i \le u_i$ and $v_i \ge v_{i+1}$ if $u_i = u_{i+1}$.

Now we give an example.

Example 3.3.7 Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & \\ 4 & 5 & \end{pmatrix}$$

be a standard d-tableau. Then $r = 5$ and applying the algorithm to A we get

$$A = A_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & \\ 4 & 5 & \end{pmatrix}, \ u_5 = 5, \ p_5 = 4 \ \xrightarrow{(1)} \ v_5 = 3, \ A_5' = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 5 & \\ 4 & & \end{pmatrix}$$

$$\xrightarrow{(2)} \ A_4 = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & & \\ 4 & & \end{pmatrix}, \ u_4 = 5, \ p_4 = 2 \ \xrightarrow{(1)} \ v_4 = 4, \ A_4' = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & \\ 3 & & \\ 4 & & \end{pmatrix}$$

$$\xrightarrow{(2)} \ A_3 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & \\ 4 & \end{pmatrix}, \ u_3 = 4, \quad p_3 = 4 \quad \xrightarrow{(1)} \ v_3 = 2, \ A_3' = \begin{pmatrix} 1 & 3 \\ 2 & 3 \\ 4 & \end{pmatrix}$$

$$\xrightarrow{(2)} A_2 = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix},\ u_2 = 3, \quad p_2 = 2 \quad \xrightarrow{(1)} v_2 = 3,\ A_2' = \begin{pmatrix} 1 & 3 \\ 2 & \end{pmatrix}$$

$$\xrightarrow{(2)} A_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ u_1 = 2,\ p_1 = 2 \xrightarrow{(1)} v_1 = 1,\ A_1' = \begin{pmatrix} 2 \end{pmatrix} \xrightarrow{(2)} A_0 = \emptyset,$$

and therefore

$$\phi(A) = \begin{pmatrix} 2 & 3 & 4 & 5 & 5 \\ 1 & 3 & 2 & 4 & 3 \end{pmatrix}.$$

In order to prove that the correspondence defined above has an inverse we define a map ψ between the set of the two-line arrays of positive integers

$$U = \begin{pmatrix} u_1 & & u_r \\ v_1 & \dots & v_r \end{pmatrix}$$

which satisfies the conditions $u_1 \leq \cdots \leq u_r$, $v_i \leq u_i$ and $v_i \geq v_{i+1}$ if $u_i = u_{i+1}$, and the set of standard d-tableaux. We define recursively a sequence of standard d-tableaux $A_1, \dots, A_r$ and a sequence of even integers $p_1, \dots, p_r$ such that the entries of A_i are the elements $u_1, \dots, u_i, v_1, \dots, v_i$, the last element in the p_ith row is u_i, and all the elements of A_i which coincides with u_i are in the first p_i rows.

Set

$$A_1 = \begin{pmatrix} v_1 \\ u_1 \end{pmatrix},$$

and for all $i = 1, \dots, r-1$ we proceed as follows

(1) Apply INSERT to input (A_i, v_{i+1}) and let $(A^*_{i+1}, p_{i+1} - 1)$ be the output.

(2) Add u_{i+1} from the tableau A^*_{i+1} to the end of the $(p_i + 1)$th row to get A_{i+1}.

Remark that the row index $p_{i+1} - 1$ is odd since A_i is a standard d-tableau, and let us say $p_{i+1} = 2s_{i+1} + 1$. If the shape of A_i is $r_1, r_1, \dots, r_q, r_q$, then the shape of A^*_{i+1} is $r_1, r_1, \dots, r_{s_{i+1}} + 1, r_{s_{i+1}}, \dots, r_q, r_q$.

At the end we define $\psi(U) = A_r$.

In order to prove that the algorithm is well-defined we have to check that step (2) gives rise to a standard d-tableau and that the elements of A_{i+1} which coincide with u_{i+1} are in the first p_{i+1} rows. It is sufficient to show that all the elements in the tableau A^*_{i+1} which are equal to u_{i+1} are in the first $p_{i+1} - 1$ rows. When $u_i < u_{i+1}$, this is obvious since the entries of A_i are less than or equal to u_i and v_{i+1} is in the first row of A^*_{i+1}. When $u_i = u_{i+1}$, we know that $v_i \geq v_{i+1}$. Furthermore, if we apply DELETE to the input (A_i, p_i), we get the output (A_i', v_i), where the tableau A_i' is obtained from A_{i-1} by adding u_i at the end of the $(p_i - 1)$th row. Then we apply INSERT to (A_i, v_{i+1}) and we get $(A^*_{i+1}, p_{i+1}-1)$. By Theorem 3.3.6 we obtain $p_{i+1}-1 > p_i$. By induction we know that all the elements in the tableau A_i which are equal to u_i are in the first p_i rows, hence all the elements in the tableau A^*_{i+1} which are equal to u_i are in the first p_{i+1} rows. Therefore all the elements in the tableau A^*_{i+1} which are equal to u_{i+1} are in the first p_{i+1} rows.

Theorem 3.3.8 *The maps ϕ and ψ are inverse to each other.*

Proof. Let A be a standard d-tableau and

$$\phi(A) = U = \begin{pmatrix} u_1 & \dots & u_r \\ v_1 & \dots & v_r \end{pmatrix}.$$

By induction on r we can prove that $\psi(U) = A$. The case $r = 1$ is trivial, and suppose $r > 1$. Let A_{r-1} be the standard d-tableau defined recursively in the definition of ϕ. Then

$$\phi(A_{r-1}) = U_{r-1} = \begin{pmatrix} u_1 & & u_{r-1} \\ v_1 & & v_{r-1} \end{pmatrix}.$$

By induction we get that $\psi(U_{r-1}) = A_{r-1}$. Note now that the steps (1) and (2) in the definition of ψ and the steps (2*) and (1*) in the definition of ϕ are, respectively, inverse to each other, and we obtain that $\psi(U) = A$. Analogously we prove that $\phi(A) = U$. □

As in the generic case, we have found a one-to-one correspondence between standard d-tableaux and monomials of the set of indeterminates $X = \{X_{ij} : i, j = 1, 2, \dots,\ 1 \le i \le j\}$. We call it the Knuth-Robinson-Schensted (KRS for short) correspondence for standard d-tableaux. Note that every monomial $\prod_{i \le j} X_{ij}^{k_{ij}}$ can be uniquely rewritten in the form $X_{v_1u_1} \cdots X_{v_ru_r}$ with $u_1 \le \cdots \le u_r$, $v_i \le u_i$ and $v_i \ge v_{i+1}$ if $u_i = u_{i+1}$. Therefore we have a one-to-one correspondence between the set of monomials in X and the set of two-line arrays of positive integers with the above mentioned properties.

We can identify a product of minors of a generic symmetric matrix $X = (X_{ij})$ with a d-tableau, that is, whenever M is a product of minors $M = M_1 \cdots M_u$ with $M_i = [a_{i1} \dots a_{ir_i} | b_{i1} \dots b_{ir_i}]$ an r_i-minor, $r_1 \ge \cdots \ge r_u$, we consider M as being the d-tableau A, where $a_{i1}, \dots, a_{ir_i}$ is the $(2i-1)$th row and $b_{i1}, \dots, b_{ir_i}$ is the $2i$th row of A. It is obvious that the standard monomials correspond to the standard d-tableaux.

If we consider only standard d-tableaux and monomials whose entries, respectively indeterminates come from an $n \times n$ generic symmetric matrix X, we get the following:

Theorem 3.3.9 KRS *is a one-to-one correspondence between the set of standard d-tableaux on the set $\{1, \dots, n\}$ and the monomials in $K[X]$.*

In the application of KRS to find Gröbner bases of determinantal ideals we need the following result.

Lemma 3.3.10 *Let A be a standard d-tableau, and $a_1, \dots, a_{r_1}$ its first row. If we have* $\mathrm{KRS}(A) = X_{v_1u_1} \cdots X_{v_ru_r}$*, then for every $s = 1, \dots, r_1$ there exists a factor $X_{v_{\alpha_1}u_{\alpha_1}} \cdots X_{v_{\alpha_s}u_{\alpha_s}}$ of* $\mathrm{KRS}(A)$ *such that $v_{\alpha_1} < \cdots < v_{\alpha_s}$ and $u_{\alpha_1} < \cdots < u_{\alpha_s}$.*

Proof. Let A_i, $i = 1, \dots, r$, be the standard d-tableaux occurring in the above described procedure of computing $\phi(A)$. The proof goes by induction on r. The case $r = 1$ is trivial and suppose that $r > 1$. The element v_r is necessarily an element of the first row of A, say $v_r = a_j$, $1 \le j \le r_1$. We have

$$\mathrm{KRS}(A_{r-1}) = \begin{pmatrix} u_1 & \dots & u_{r-1} \\ v_1 & & v_{r-1} \end{pmatrix}$$

and the first row of A_{r-1} differs from that of A only in position j. Now we have to distinguish two cases. If $s \ne j$, then by induction there is a sequence $\alpha_1 < \cdots < \alpha_s$ such that $v_{\alpha_1} < \cdots < v_{\alpha_s} = a_s$. If $s = j$, then we can consider $\alpha_1 = r$ in case $j = 1$, and $\alpha_1 < \cdots < \alpha_{j-1} < \alpha_j = r$ with $v_{\alpha_1} < \cdots < v_{\alpha_{j-1}} = a_{j-1}$ in case $j \ne 1$. Obviously we must have $u_{\alpha_1} < \cdots < u_{\alpha_s}$. □

(**A**) Throughout this part we shall use the original Knuth-Robinson-Schensted correspondence defined by Knuth in [51]. In the pfaffian case, the Knuth-Robinson-Schensted correspondence has some important particularities. To be more specific, let $\nu = \pi_1 \cdots \pi_r$ be a standard monomial, π_i a $2s_i$-pfaffian; one can consider ν as a standard tableau (it means that the ith column of the tableau has $2s_i$ elements, $s_1 \ge \cdots \ge s_r$, the numbers in each row are in non-decreasing order from left to right, and the numbers in each column are in strictly increasing order from top to bottom) of shape $s_1, \dots, s_r$, and then

$$\mathrm{KRS}(\nu) = \begin{pmatrix} u_1 & \dots & u_{2s} \\ v_1 & & v_{2s} \end{pmatrix},$$

where $s = s_1 + \cdots + s_r$, and the pairs (u_i, v_i) are arranged in non-decreasing lexicographic order from left to right. We should observe that in the above two-line array do not appear pairs (u_i, v_i) with $u_i = v_i$, and once with a pair (u_i, v_i) appears the pair (v_i, u_i), too. With this array we associate a monomial $M = X_{a_1b_1} \cdots X_{a_sb_s}$ in $K[X]$, such that the sequence $b = (b_1, \dots, b_s)$ is the sequence of those v_i for $u_i < v_i$, and a_i is the corresponding element u_i.

Define the width of a sequence to be the length of the longest decreasing subsequence; one observes that $\mathrm{width}(b) = 1/2\mathrm{width}(v)$. Let $b_{i_1} > \cdots > b_{i_r}$ be a longest decreasing subsequence of b. If we take into account the relationship between the sequences v and b, we conclude that there exists a decreasing subsequence $v_{j_1} > \cdots > v_{j_r}$ of v that can be extended to a subsequence of length $2r$ by gluing it to the subsequence $u_{j_r} > \cdots > u_{j_1}$. On the other hand, if $v_{i_1} > \cdots > v_{i_{2r}}$ is a longest decreasing subsequence of v, then $u_{i_r} < v_{i_r}$ and $u_{i_{r+1}} > v_{i_{r+1}}$, otherwise we get decreasing subsequences of v of length greater than $2r$. So, the sequence $v_{i_1} > \cdots > v_{i_r}$ is a decreasing subsequence of b of length r.

The next step is to show that between the sequences v and b there exists a deeper connection, namely that the shape of $P(b) = \text{INSERT}(b)$, the tableau obtained by performing algorithm INSERT on the sequence b, is just $s_1, \ldots, s_r$. The following result represents the first application of Greene's theorem to KRS.

Proposition 3.3.11 *Let ν be a standard monomial of shape $(s_1, \ldots, s_r)$, and let $b = (b_1, \ldots, b_s)$ be the sequence obtained by performing* KRS *on ν. If $P(b) = \text{INSERT}(b)$, then the shape of $P(b)$ is $(s_1, \ldots, s_r)$.*

Proof. Let us recall that

$$\text{KRS}\,(\mu) = \begin{pmatrix} u_1 & & u_{2s} \\ v_1 & & v_{2s} \end{pmatrix},$$

where $s = s_1 + \cdots + s_r$, and the pairs (u_i, v_i) are arranged in non-decreasing lexicographic order from left to right. We break the sequence $v = (v_1, \ldots, v_{2s})$ into two subsequences $b = (v_i : u_i < v_i)$, $c = (v_i : u_i > v_i)$, and their terms are written in the order inherited from v. To the $2 \times s$ matrix

$$\begin{pmatrix} u_i \\ b_i \end{pmatrix},$$

one associates the matrix $B = (b_{ij})$, where b_{ij} is the number of occurrences of the pair (i, j) in the matrix above. Analogously, to the $2 \times s$ matrix

$$\begin{pmatrix} u_i \\ c_i \end{pmatrix},$$

one associates the matrix $C = (c_{ij})$. Obviously, the matrix C is the transpose of B. Let $(P'|Q')$ be the standard bitableau corresponding to the matrix B in the KRS correspondence. Then to the matrix C has to correspond the standard bitableau $(Q'|P')$; see Knuth [51, Theorem 3]. We get that $P(b) = P'$ and $P(c) = Q'$, so shape $P(b) =$ shape $P(c)$. Recall that $d_k(v)$ be the length of the longest subsequence of v which has no increasing subsequences of length $k+1$. From Theorem 3.2.2 we get that $d_k(v) = 2s_1 + \cdots + 2s_k$. If one denotes shape $P(b) = s'_1, \ldots, s'_q$, then $d_k(b) = d_k(c) = s'_1 + \cdots + s'_q$. It is obvious that $2d_k(b) \leq d_k(v)$. We prove that the converse is also true. If we take a subsequence of v which has no increasing subsequences of length $k+1$, this should be the union of k decreasing subsequences. These subsequences break up into two decreasing subsequences, one of them is a subsequence of b and the other one of c. So, we get a subsequence of b and a subsequence of c which are unions of at most k decreasing subsequences, therefore $d_k(b) + d_k(c) \geq d_k(v)$. This proves that $2d_k(b) = d_k(v)$, and therefore shape $P(b) = (s_1, \ldots, s_r)$. □

3.4 Gröbner bases of determinantal ideals

We start by recalling the definitions and some of the main properties of Gröbner bases, initial ideals and initial algebras. The interested reader to know more on this topics is referred to Eisenbud [35] for the theory of Gröbner bases, respectively to Conca, Herzog and Valla [27] for initial algebras.

Let K be a field and let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over K in the indeterminates $X_1, \ldots, X_n$. A *monomial* m of R is a product of indeterminates $m = \prod X_i^{k_i}$ with $k_i \in \mathbb{N}$. A *term* of R is an element of the form λm with $\lambda \in K$, $\lambda \neq 0$ and m a monomial. We consider a total order τ, denoted by $<_\tau$, on the set of all the monomials in R. Then τ is a *monomial order* if it satisfies the following conditions:

(a) $1 <_\tau m$ for all monomials different from 1.

(b) If $m_1 <_\tau m_2$, then $m_1 m_3 <_\tau m_2 m_3$ for all monomials m_1, m_2, m_3 in R.

Let us fix a monomial order τ on the set of monomials in R. For a polynomial $f \in R$ we define $\operatorname{in}_\tau(f)$, the *initial monomial* of f with respect to τ, as being the greatest monomial in the (unique) representation of f as a sum of terms in R. If I is an ideal of R, then $\operatorname{in}_\tau(I)$ the *initial ideal* of I is the ideal of R generated by the monomials $\operatorname{in}_\tau(f)$ for all $f \in I$. A subset F of I is called a *Gröbner basis* of I with respect to τ whether the initial ideal $\operatorname{in}_\tau(I)$ equals the ideal of R generated by $\operatorname{in}_\tau(f)$ with $f \in F$. If A is a K-subalgebra of R, then $\operatorname{in}_\tau(A)$ the *initial algebra* of A is the K-subalgebra of R generated by the monomials $\operatorname{in}_\tau(f)$ for all $f \in I$. Obviously $\operatorname{in}_\tau(A)$ is the semigroup ring $K[S]$ of a suitable subsemigroup S of $\mathbb{N}^n$.

(**G**) As we mentioned we identify a product of minors of a generic matrix $X = (X_{ij})$ with a bitableau, that is, whenever $\mu = \delta_1 \cdots \delta_u$ with $\delta_i = [a_{i1}, \ldots, a_{ir_i} | b_{i1}, \ldots, b_{ir_i}]$ an r_i-minor, $r_1 \geq \cdots \geq r_u$, we consider μ as being the bitableau $\Sigma = (A, B)$, where $A = (a_{ij})$ and $B = (b_{ij})$ are tableaux. It is obvious that the standard monomials correspond to the standard bitableaux. In this part restrict our attention to standard bitableaux and monomials whose entries, respectively indeterminates come from an $m \times n$ generic matrix X.

With this convention, the straightening law on $K[X]$ can be formulated as follows

Theorem 3.4.1 (a) *The standard bitableaux form a K-vector space basis of $K[X]$.*
(b) *The product of two minors $\delta_1, \delta_2 \in \Delta(X)$ such that $\delta_1\delta_2$ is not a standard bitableau has a representation*

$$\delta_1\delta_2 = \sum \lambda_i \xi_i \eta_i, \quad \lambda_i \in K, \quad \lambda_i \neq 0,$$

where $\xi_i\eta_i$ is a standard bitableau and $\xi_i \prec \delta_1$, $\delta_2 \prec \eta_i$ (we allow here that η_i is the empty minor).
(c) *The standard representation of an arbitrary bitableau Σ, i.e. its representation as a K-liniar combination of standard bitableaux, can be found by successive applications of the straightening relations in* (b).

Let $\delta \in \Delta(X)$ and $I(X,\delta)$ the ideal of $K[X]$ cogenerated by δ. From Theorem 3.4.1 we deduce:

Corollary 3.4.2 *The set of all standard bitableaux Σ with* $\min(\Sigma) \notin \Delta(X,\delta)$ *is a K-basis of $I(X,\delta)$.*

Set $\delta = [a_1,\ldots,a_r|b_1,\ldots,b_r]$. For systematic reasons it is convenient to set $a_{r+1} = m+1$ and $b_{r+1} = n+1$. Let J_δ be the set of all t-minors $[a'_1,\ldots,a'_t|b'_1,\ldots,b'_t]$, $t = 1,\ldots,r+1$ which satisfy the conditions $a'_i \geq a_i$, $b'_i \geq b_i$ for $i = 1,\ldots,t-1$, and $a'_t < a_t$ or $b'_t < b_t$.

Lemma 3.4.3 *The set J_δ is a minimal system of generators of $I(X,\delta)$.*

Proof. Let $\gamma = [a'_1,\ldots,a'_s|b'_1,\ldots,b'_s]$ be an arbitrary minor with the property that $\gamma \notin \Delta(X,\delta)$. Then there exists an integer $t \leq \min\{s, r+1\}$ such that $a'_t < a_t$ or $b'_t < b_t$. Now we expand γ with respect to the last $s-t$ rows. So we may write it as a linear combination of products of $(s-t)$-minors and t-minors of J_δ. Therefore J_δ is a system of generators of $I(X,\delta)$. If J_δ is not a minimal system of generators, then there exists $\gamma' \in J_\delta$ such that $\gamma' = \sum f_i\delta_i$ with $\gamma' \neq \delta_i \in J_\delta$ and $f_i \in K[X]$. By the straightening law on $K[X]$ we may suppose that f_i are standard monomials. Since γ' is a standard bitableau it must appear in the representation of some $f_i\delta_i$ as a linear combination of standard bitableaux. In this case, by Theorem 3.4.1, we must have $\gamma' < \delta_i$. By definition any two elements of J_δ of different size are incomparable. Therefore, $\deg(\gamma') = \deg(\delta_i)$. So we obtain that $\deg(f_i) = 0$ and $\gamma' = \delta_i$, a contradiction. □

From now on τ will be a diagonal term order on the polynomial ring $K[X]$, i.e. the initial monomial of a minor $\delta = [a_1,\ldots,a_r|b_1,\ldots,b_r]$ is $\mathrm{in}_\tau(\delta) =$

$\prod_{i=1}^{r} X_{a_i b_i}$, the product of the elements on the main diagonal of δ. For instance, we can consider the lexicographic order induced by the variable order

$$X_{11} > X_{12} > \cdots > X_{1n} > X_{21} > X_{22} > \cdots > X_{2n} > \cdots > X_{m1} > \cdots > X_{mn}.$$

In the following we show that the set J_δ is a Gröbner basis for $I(X,\delta)$ with respect to the term order τ. This result was first published by Herzog and Trung [45] and its proof was inspired by Sturmfels [61] who has proved, by using KRS, that the set of all r-minors is a Gröbner basis for $I_r(X)$ (with respect to a diagonal term order). In fact, Sturmfels has observed that for an ideal I of $K[X]$ which has a K-basis of standard bitableaux, KRS(I) is a K-vector subspace of $K[X]$ that has two of the properties of an initial ideal: it has a monomial basis and its Hilbert function is the same as the Hilbert function of I. If it happens that KRS(I) is contained (or contains) in $\text{in}_\tau(I)$, then we must have equality by the Hilbert function. To be precise, we have the following result (see Bruns and Conca [16], and Sturmfels [61]):

Lemma 3.4.4 *Let I be an ideal of $K[X]$ that has a K-basis $\mathcal{B}$ of standard bitableaux (monomials), and let $\mathcal{S}$ be a subset of I. Assume that for all $\Sigma \in \mathcal{B}$ there exists $s \in \mathcal{S}$ such that* $\text{in}_\tau(s) \mid \text{KRS}(\Sigma)$. *Then $\mathcal{S}$ is a Gröbner basis of I and* $\text{in}_\tau(I) = \text{KRS}(I)$.

Proof. Let J be the ideal generated by the monomials $\text{in}_\tau(s)$ with $s \in \mathcal{S}$. By hypothesis we have that $\text{KRS}(I) \subseteq J \subseteq \text{in}_\tau(I)$. But KRS$(I)$ and I have the same Hilbert function since KRS is degree preserving. But it is well-known that $\text{in}_\tau(I)$ and I have the same Hilbert function. It follows that $\text{KRS}(I) = J = \text{in}_\tau(I)$. □

Now we use Lemma 3.4.4 to prove:

Theorem 3.4.5 *The set J_δ is a Gröbner basis for $I(X,\delta)$ with respect to the term order τ.*

Proof. As we already know from Corollary 3.4.2, the set $\mathcal{B}$ of standard bitableaux Σ with the property that $\min(\Sigma) = [a_{11},\ldots,a_{1r_1}|b_{11},\ldots,b_{1r_1}]$ is not greater or equal than δ is a K-basis of $I(X,\delta)$. Set $\mathcal{S} = J_\delta$ and $\text{KRS}(\Sigma) = X_{u_1v_1}\cdots X_{u_pv_p}$. In the view of Lemma 3.4.4, we have to show that for all $\Sigma \in \mathcal{B}$ there exists $s \in \mathcal{S}$ such that $\text{in}_\tau(s) \mid \text{KRS}(\Sigma)$. The condition $\min(\Sigma) \notin \Delta(X,\delta)$ implies that either $r_1 > r$ or $r_1 \le r$ and $a_{1w} < a_w$ or $b_{1w} < b_w$ for some $w = 1,\ldots,r_1$. In the first case, by Theorem 3.2.1 we can find an increasing subsequence of $v = (v_1,\ldots,v_p)$ of length r_1. Now it is easily to find an r_1-minor $s \in S$ with the desired property. In the second case, we apply Lemma 3.3.4 to get a factor $X_{u_{\alpha_1}v_{\alpha_1}}\cdots X_{u_{\alpha_w}v_{\alpha_w}}$ of KRS(Σ) such that $u_{\alpha_1} < \cdots < u_{\alpha_w}$,

$v_{\alpha_1} < \cdots < v_{\alpha_w}$ and $u_{\alpha_w} = a_{1w}$. We can take $s = [u_{\alpha_1}, \cdots, u_{\alpha_w} | v_{\alpha_1}, \cdots, v_{\alpha_w}]$ and obviously $s \in \mathcal{S}$. □

In particular we have:

Corollary 3.4.6 *The set of all t-minors is a Gröbner basis for $I_t(X)$ with respect to a diagonal term order.*

(S) Let $X = (X_{ij})$ be a generic symmetric matrix. Similar to the generic case we identify a product of minors with a d-tableau, that is, whenever $M = M_1 \cdots M_u$ with $M_i = [a_{i1} \ldots a_{ir_i} | b_{i1}, \ldots, b_{ir_i}]$ an r_i-minor, $r_1 \geq \cdots \geq r_u$, we consider M as being the d-tableau A, where $a_{i1}, \ldots, a_{ir_i}$ is the $(2i-1)$th row and $b_{i1}, \ldots, b_{ir_i}$ is the $2i$th row of A. In the following we consider only standard d-tableaux and monomials whose entries, respectively indeterminates come from an $n \times n$ generic symmetric matrix X. In this frame, the straightening law on $K[X]$ can be formulated as follows:

Theorem 3.4.7 (a) *The standard d-tableaux form a K-base of $K[X]$.*
(b) *Let $M_i = [a_{i1}, \ldots, a_{ir_i} | b_{i1}, \ldots, b_{ir_i}] \in \Delta(X)$, $i = 1, \ldots, s$ such that $M = M_1 \cdots M_s$ is not a standard d-tableau. Then M has a representation $M = \sum \lambda_i N_i$, $\lambda_i \in K$, $\lambda_i \neq 0$, where N_i are standard d-tableaux which satisfy the following condition: if* $\min(N_i) = [c_{i1}, \ldots, c_{is_i} | d_{i1}, \ldots, d_{is_i}]$ *and we set $c_i = \{c_{i1}, \ldots, c_{is_i}\}$, $d_i = \{d_{i1}, \ldots, d_{is_i}\}$, $a_i = \{a_{i1}, \ldots, a_{ir_i}\}$ and $b_i = \{b_{i1}, \ldots, b_{ir_i}\}$, then in the lexicographic order on H the sequence $c_1, d_1, \ldots, c_r, d_r$ is less than or equal to every sequence obtained by permuting the elements $a_1, b_1, \ldots, a_s, b_s$.*
(c) *The standard representation of an arbitrary d-tableau, i.e. its representation as a K-liniar combination of standard d-tableaux, can be found by successive applications of the straightening relations in* (b).

Let $\alpha \in H$, $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ and $I(X, \alpha)$ the ideal of $K[X]$ cogenerated by α. From Theorem 3.4.7 we deduce:

Corollary 3.4.8 *The set of all standard d-tableaux A with* $\min(A) \notin \Delta(X, \alpha)$ *is a K-basis of $I(X, \alpha)$.*

As before, we set $\alpha_{r+1} = n+1$ and let J_α be the set of all the doset t-minors $[a_1, \ldots, a_t | b_1, \ldots, b_t]$, $t = 1, \ldots, r+1$ which satisfy the conditions $a_i \geq \alpha_i$, for $i = 1, \ldots, t-1$, and $a_t < \alpha_t$.

Lemma 3.4.9 *The set J_α is a minimal system of generators of $I(X, \alpha)$.*

Proof. Let start with an observation. If $M = [a_1, \ldots, a_s | b_1, \ldots, b_s]$ is a doset minor of X and we expand M with respect to the last row, we may write it as a linear combination, with polynomial coefficients, of the doset minors

$[a_1,\ldots,a_{s-1}|b_1,\ldots,\hat{b}_j,\ldots,b_s]$, $j=1,\ldots,s$ (here ^ means that the corresponding index is missing).

Let $M=[a_1,\ldots,a_s|b_1,\ldots,b_s]$ be an arbitrary doset minor with the property that $M\notin\Delta(X,\alpha)$. Then there exists an integer $t\le\min\{s,r+1\}$ such that $a_t<\alpha_t$ and $a_i\ge\alpha_i$ if $i<t$. Now we expand M with respect to the last $s-t$ rows. The above observation says that we can write M as a linear combination of doset t-minors of J_α. So J_α is a system of generators of $I(X,\alpha)$.

If J_α is not a minimal system of generators, then there exists $M'\in J_\alpha$ such that $M'=\sum g_iM_i$ with $M'\neq M_i\in J_\alpha$ and $g_i\in K[X]$. By the straightening law on $K[X]$ we may assume that g_i are standard monomials. Because M' is a standard d-tableau it must appear in the representation of some g_iM_i as a linear combination of standard d-tableaux. Now, by Theorem 3.4.7, we must have that the sequence of row indices of M' is less than or equal to that of M_i. On the other hand, any two elements of J_α of different size have incomparable sequences of row indices. Therefore, $\deg(M')=\deg(M_i)$. So we obtain that $\deg(g_i)=0$ and $M'=M_i$, and this is impossible. □

One considers again a diagonal term order τ on the polynomial ring $K[X]$. Thus the initial monomial of a doset minor $M=[a_1,\ldots,a_r|b_1,\ldots,b_r]$ is $\mathrm{in}_\tau(M)=\prod_{i=1}^r X_{a_ib_i}$, the product of the elements on the main diagonal of M. There are various choices of τ. For instance, we can consider the lexicographic order induced by the variable order

$$X_{11}>X_{12}>\cdots>X_{1n}>X_{22}>\cdots>X_{2n}>\cdots>X_{n-1n}>\cdots>X_{nn}.$$

Our aim is to show that the set J_α is a Gröbner basis for $I(X,\alpha)$ with respect to the term order τ. This result was proved by Conca [22] and it relies on Lemma 3.4.4, too.

Theorem 3.4.10 *The set J_α is a Gröbner basis for $I(X,\alpha)$ with respect to the term order τ.*

Proof. From Corollary 3.4.8 the set $\mathcal{B}$ of standard d-tableaux A with the property that $\min(A)=\{a_1,\ldots,a_t\}$ is not greater or equal than α is a K-basis of $I(X,\alpha)$.

Set $\mathcal{S}=J_\delta$ and $\mathrm{KRS}(A)=X_{u_1v_1}\cdots X_{u_pv_p}$. In the view of Lemma 3.4.4, we have to show that for $A\in\mathcal{B}$ there exists $s\in\mathcal{S}$ such that $\mathrm{in}_\tau(s)|\,\mathrm{KRS}(A)$. The condition $\min(A)\notin\Delta(X,\alpha)$ means that there exists w, $w=1,\ldots,r+1$, such that $a_w<\alpha_w$ and we choose w minimal with respect to this property. By using Lemma 3.3.10 we get a factor $X_{u_{\alpha_1}v_{\alpha_1}}\cdots X_{u_{\alpha_w}v_{\alpha_w}}$ of $\mathrm{KRS}(A)$ such that $u_{\alpha_1}<\cdots<u_{\alpha_w}$, $v_{\alpha_1}<\cdots<v_{\alpha_w}$ and $v_{\alpha_w}=a_w$. We can take $s=[u_{\alpha_1},\ldots,u_{\alpha_w}|v_{\alpha_1},\ldots,v_{\alpha_w}]$ and obviously $s\in\mathcal{S}$. □

In particular we have:

Corollary 3.4.11 *The set of all doset t-minors is a Gröbner basis for $I_t(X)$ with respect to a diagonal term order.*

(**A**) In this part we apply the same methods to deal with ideals generated by pfaffians of a fixed size. As in the previous section we identify a product of pfaffians of a generic alternating matrix $X = (X_{ij})$ with a tableau. If $\mu = \pi_1 \cdots \pi_r$ with π_i a $2s_i$-pfaffian, we consider μ as a tableau; it means that the ith column of the tableau has $2s_i$ elements, $s_1 \geq \cdots \geq s_r$, the numbers in each row are in non-decreasing order from left to right, and the numbers in each column are in strictly increasing order from top to bottom. Clearly the standard monomials correspond to the standard tableaux. In this part we confine to standard tableaux and monomials whose entries, respectively indeterminates come from an $n \times n$ generic alternating matrix X.

With this convention, the straightening law on $K[X]$ can be formulated as follows

Theorem 3.4.12 (a) *The standard tableaux form a K-vector space basis of $K[X]$.*
(b) *The product of two pfaffians $\pi_1, \pi_2 \in \Pi(X)$ such that $\pi_1\pi_2$ is not a standard tableau has a representation*

$$\pi_1\pi_2 = \sum \lambda_i \xi_i \eta_i, \quad \lambda_i \in K, \quad \lambda_i \neq 0,$$

where $\xi_i\eta_i$ is a standard tableau and $\xi_i \prec \pi_1$, $\pi_2 \prec \eta_i$ (we allow here that η_i is the empty pfaffian).
(c) *The standard representation of an arbitrary tableau P, i.e. its representation as a K-liniar combination of standard tableaux, can be found by successive applications of the straightening relations in* (b).

Let $\pi \in \Pi(X)$ and $I(X,\pi)$ the ideal of $K[X]$ cogenerated by π. From Theorem 3.4.12 we obtain

Corollary 3.4.13 *The set of all standard tableaux P with* $\min(P) \notin \Pi(X,\pi)$ *is a K-basis of $I(X,\pi)$.*

This time we consider the term order τ on the polynomial ring $K[X]$ induced by the variable order

$$X_{1n} > X_{1n-1} > \cdots > X_{12} > X_{2n} > X_{2n-1} > \cdots > X_{23} > \cdots > X_{n-1n}.$$

With respect to τ the initial monomial of a pfaffian $\pi = [a_1, \ldots, a_{2r}]$ is $\mathrm{in}_\tau(\pi) = \prod_{i=1}^{r} X_{a_i a_{2r-i+1}}$.

Unfortunately KRS does not behave so "nice" in the case of pfaffians as in the case of minors. For a given a pfaffian π, KRS does not map the standard monomial K-basis of $I(X,\pi)$ to the ideal generated by the initial terms of the pfaffians that generate $I(X,\pi)$. We record here an example which illustrates that and gives an answer to a question raised by Herzog and Trung in [45, pg.26].

Example 3.4.14 Let $n=5$, $\pi=[1245]$, and P the following standard tableau

$$P=\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & \\ 5 & \end{pmatrix}.$$

An easy computation gives

$$\mathrm{KRS}\,(P)=\begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 & 1 \end{pmatrix},$$

and this two-line array is mapped further to the monomial $X_{13}X_{15}X_{24}$. We get that the standard monomial $\nu=[1235][14]$ which belongs to a K-basis of $I(X,\pi)$ has the property that $\mathrm{KRS}\,(\nu)=X_{13}X_{15}X_{24}$ does not belong to the ideal generated by $\mathrm{in}_\tau(\xi)$, where ξ is a pfaffian which is not greater or equal than π. In fact, only two pfaffians satisfy this condition, namely [1234] and [1235], and their initial terms are $X_{14}X_{23}$ and $X_{15}X_{23}$, both of them do not dividing $\mathrm{KRS}\,(\nu)$.

In the view of Example 3.4.14 we restrict ourselves to the case of pfaffians of fixed size. Let J_t denote the set of all $2t$-pfaffians of X.

Theorem 3.4.15 *The set J_t is a Gröbner basis of $I_t(X)$ with respect to the given term order τ.*

Proof. From Corollary 3.4.13 the set $\mathcal{B}$ of standard tableaux P with the property that $\min(P)$ is not greater or equal than $\pi=[1,\dots,2t-2]$ is a K-basis of $I_t(X)$. Set $\mathcal{S}=J_t$ and $\mathrm{KRS}\,(P)=X_{a_1b_1}\cdots X_{a_pb_p}$. In the view of Lemma 3.4.4, we have to show that for $P\in\mathcal{B}$ there exists $s\in\mathcal{S}$ such that $\mathrm{in}_\tau(s)|\mathrm{KRS}\,(P)$.

The condition $\min(P)\notin\Pi(X,\pi)$ means that the length of the first column of P is greater or equal than $2t$. By Theorem 3.2.1 we obtain $\mathrm{width}\,(b)\geq t$, where $b=(b_1,\dots,b_p)$. Let $b_{i_1}>\cdots>b_{i_t}$ be a longest decreasing subsequence of b. If we take into account the relationship between the sequences v and b (see Section 3.1, part (A)), we conclude that there exists a decreasing subsequence $v_{j_1}>\cdots>v_{j_t}$ of v, respectively an increasing subsequence $u_{j_t}>\cdots>u_{j_1}$ of u. Take $s=[u_{j_1},\dots,u_{j_t},v_{j_t},\dots,v_{j_1}]\in\mathcal{S}$ and apply Lemma 3.4.4. □

Chapter 4

Powers and products of determinantal ideals

4.1 Primary decomposition of the (symbolic) powers of determinantal ideals

(**G**) In this part we describe the primary decomposition of powers and, more generally, products of determinantal ideals. Let $J = I_{t_1}(X) \cdots I_{t_u}(X)$ with $t_1 \geq \cdots \geq t_u$. Since $I_{t_i}(X)$ are prime ideals, only the ideals $I_t(X)$ with $t \leq t_1$ can be associated to J. Thus we have to find the primary components of J with respect to the ideals $I_t(X)$. The natural candidates are their symbolic powers. It turns out that the (symbolic) powers of determinantal ideals have K-bases of standard monomials. The elements of the standard monomial K-bases of $I_t(X)^{(k)}$ are described in terms of certain functions γ_t which have to be defined.

Given a sequence of numbers $\sigma = (s_1, \ldots, s_p)$ and a number t we define

$$\gamma_t(\sigma) = \sum_{i=1}^{p} \max(s_i + 1 - t, 0).$$

Then we extend the definition to products of minors μ by setting:

$$\gamma_t(\mu) = \gamma_t(\sigma)$$

where σ is the shape of μ. The functions γ_t were introduced by De Concini, Eisenbud, and Procesi [29] to describe the symbolic powers of $I_t(X)$ and the primary decomposition of the powers of $I_t(X)$. Actually, they defined the functions γ_t only for shapes (in terms of the dual shape), but it is a triviality to show that both definitions coincide for shapes.

Theorem 4.1.1 *The ideal $I_t(X)^{(k)}$ is the K-vector space generated by the standard monomials μ with $\gamma_t(\mu) \geq k$, and it contains all products of minors μ' with $\gamma_t(\mu') \geq k$. One has*

$$I_t(X)^{(k)} = \sum_{\sigma=(s_1,\dots,s_p),\ \gamma_t(\sigma)\geq k} I_{s_1}(X)\cdots I_{s_p}(X).$$

Of course, the above description of $I_t(X)^{(k)}$ contains only finitely many summands. The simplest non-trivial case is $t = k = 2$ when we get $I_2(X)^{(2)} = I_2(X)^2 + I_3(X)$. If $m = 2$ or $n = 2$, then $I_2(X)^{(2)} = I_2(X)^2$. This phenomenon occurs for all ideals of maximal minors.

Corollary 4.1.2 *Suppose that $m \leq n$. Then the symbolic powers of the ideal $I_m(X)$ of maximal minors coincide with the ordinary ones.*

Proof. A monomial μ has $\gamma_m(\mu) \geq k$ if and only if its first k factors have size exactly m. □

None of the results so far depends on the characteristic of field K. In the words of Bruns and Vetter, "quite surprisingly, the primary decomposition of products $I_{t_1}(X)\cdots I_{t_u}(X)$, in particular of powers $I_t(X)^k$, can not be given without reference to the characteristic". For instance, in characteristic 2 one has $I_1(X)I_3(X) \not\subseteq I_2(X)^2$ provided $m, n \geq 4$; see [21, Remarks (10.14)]. The next theorem was proved by De Concini, Eisenbud, and Procesi [29] in characteristic 0, and generalized later by Bruns and Vetter [21, Theorem (10.9)].

Theorem 4.1.3 *Let $\sigma = (t_1, \dots, t_u)$ be a non-increasing sequence of integers and suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t_i, m-t_i, n-t_i)$ *for $i = 1, \dots, u$. Then*

$$I_{t_1}(X)\cdots I_{t_u}(X) = \bigcap_{i=1}^{t_1} I_i(X)^{(\gamma_i(\sigma))}$$

is a primary decomposition. In particular, $I_{t_1}(X)\cdots I_{t_u}(X)$ is generated by the standard monomials μ with $\gamma_i(\mu) \geq \gamma_i(\sigma)$ for all $i = 1, \dots, t_1$, and it contains all products of minors μ' satisfying these conditions.

If we restrict to ordinary powers of determinantal ideals we have the following:

Corollary 4.1.4 *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t, m-t, n-t)$. *Then*

$$I_t(X)^k = \bigcap_{i=1}^{t} I_i(X)^{(k(t+1-i))}$$

is a primary decomposition. In particular, $I_t(X)^k$ *is generated by the standard monomials* μ *with* $\gamma_i(\mu) \geq k(t+1-i)$ *for all* $i = 1, \ldots, t$*, and it contains all products of minors* μ *satisfying these conditions. Moreover this primary decomposition is irredundant if* $t < \min(m,n)$ *and* $k \geq (w-1)/(w-t)$ *where* $w = \min(m,n)$.

Note that the primary decomposition of $I_t(X)^k$ is irredundant for $k \gg 0$ provided $t < \min(m,n)$. Furthermore, independently of the characteristic the right side of the equality in Corollary 4.1.4 is the integral closure of $I_t(X)^k$; see Bruns [15, Theorem (1.3)].

De Concini, Eisenbud, and Procesi [29] also introduced another class of functions associated to a sequence of numbers $\sigma = (s_1, \ldots, s_p)$ defined by

$$\beta_t(\sigma) = \sum_{i=1}^{t} s_i.$$

(Actually, they also defined the functions β_t only for shapes.) We extend the definition to products of minors μ by setting:

$$\beta_t(\mu) = \beta_t(\sigma)$$

where σ is the shape of μ.

Each class of these functions defines a partial order on the set of sequences of integers as follows: for σ, λ two such sequences we define

$$\sigma \geq_\beta \lambda \text{ if and only if } \beta_i(\sigma) \geq \beta_i(\lambda) \text{ for all } i,$$

$$\sigma \geq_\gamma \lambda \text{ if and only if } \gamma_i(\sigma) \geq \gamma_i(\lambda) \text{ for all } i.$$

When σ and λ are shapes, it turns out that both partial orders coincide; see De Concini, Eisenbud, and Procesi [29, Prop. 1.1].

Proposition 4.1.5 *Let* σ *and* λ *be two shapes. One has* $\sigma \geq_\beta \lambda$ *if and only if* $\sigma \geq_\gamma \lambda$.

Proposition 4.1.5 suggests that we can replace the γ-functions by the β-functions in the description of the standard bases of products and powers of determinantal ideals. For instances, for powers (the case we are interested most) we can state:

Corollary 4.1.6 *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t, m-t, n-t)$. *Then the ideal* $I_t(X)^k$ *has a* K*-basis consisting of all standard monomials* μ *with* $\beta_k(\mu) \geq kt$.

Corollary 4.1.6 follows readily from the next lemma.

Lemma 4.1.7 *Let λ be a shape and k, t positive integers. Then $\gamma_i(\lambda) \geq k(t-i+1)$ for all $i = 1, \dots, t$ if and only if $\beta_k(\lambda) \geq kt$.*

Proof. Let $\sigma = (t, \dots, t)$ be a shape such that t appears of k times. Then $\gamma_i(\lambda) \geq k(t-i+1)$ if and only if $\gamma_i(\lambda) \geq \gamma_i(\sigma)$, and by 4.1.5 we get $\beta_i(\lambda) \geq \beta_i(\sigma)$ for all i. Now one takes $i = k$ in the last inequality and the necessity follows. For sufficiency we have to show that $\beta_k(\lambda) \geq kt$ if and only if $\beta_i(\lambda) \geq \beta_i(\sigma)$ for all $i \leq k$, that is $\beta_i(\lambda) \geq it$ for all $i \leq k$. Set $\lambda = (\lambda_1, \dots, \lambda_u)$ and suppose that there exist an $i \leq k$ such that $\lambda_1 + \dots + \lambda_{i-1} < (i-1)t$ and $\lambda_1 + \dots + \lambda_i \geq it$. Then $\lambda_1 + \dots + \lambda_{i-1} + \lambda_i < (i-1)t + \lambda_i$, therefore $\lambda_i > t$. On the other hand, $\lambda_1 \geq \dots \geq \lambda_i$ and so we obtain $\lambda_1 + \dots + \lambda_{i-1} > (i-1)t$, a contradiction. □

(**S**) Let $X = (X_{ij})$ be a generic symmetric matrix over a field K of characteristic 0. Set again $J = I_{t_1}(X) \cdots I_{t_u}(X)$ with $t_1 \geq \dots \geq t_u$. As in part (G), we find out that the primary components of J are the symbolic powers of the ideals $I_t(X)$, $t \leq t_1$. It happens again that the (symbolic) powers of determinantal ideals have K-bases of standard monomials. The elements of the standard monomial K-bases of $I_t(X)^{(k)}$ are also described in terms of the functions γ_t; see Abeasis [1].

Theorem 4.1.8 *The ideal $I_t(X)^{(k)}$ is the K-vector space generated by the standard monomials M with $\gamma_t(M) \geq k$, and it contains all products of minors M' with $\gamma_t(M') \geq k$. One has*

$$I_t(X)^{(k)} = \sum_{\sigma=(s_1,\dots,s_p),\ \gamma_t(\sigma)\geq k} I_{s_1}(X) \cdots I_{s_p}(X).$$

Remark that for $n = 3$ we obtain $I_2(X)^{(2)} = I_2(X)^2 + I_3(X)$. Thus we can not get a similar result to Corollary 4.1.2 for $I_{n-1}(X)$, the ideal of $K[X]$ generated by all maximal minors of X.

(**A**) For an $n \times n$ generic alternating matrix X we get similar results with respect to the primary decomposition of products (powers) of pfaffian ideals. First we show that the symbolic powers of pfaffian ideals have K-bases of standard monomials, whose elements we also describe in terms of the functions γ_t. Recall that $I_{t+1}(X)$ denotes the ideal generated by all the $(2t+2)$-pfaffians of X.

Next lemma is the key of the inductive proofs of results involving pfaffians.

Lemma 4.1.9 *Let $X = (X_{ij})$, $Y = (Y_{ij})$ be generic alternating matrices having sizes $n \times n$ and $(n-2) \times (n-2)$ respectively. Then the substitutions*

$$Y_{ij} \longmapsto X_{i+2j+2} + (X_{jn-1}X_{in} - X_{in-1}X_{jn})X_{n-1n}^{-1}, \quad i,j = 1,\ldots,n-2$$

yield a ring isomorphism

$$K[Y, X_{1n-1}, \ldots, X_{n-2n-1}, X_{1n}, \ldots, X_{n-1n}][X_{n-1n}^{-1}] \cong K[X, X_{n-1n}^{-1}],$$

which map the extension of $I_t(Y)$ to the extension of $I_{t+1}(X)$, $t \geq 1$. In particular, it induces an isomorphism

$$P_t(Y)[X_{1n-1}, \ldots, X_{n-2n-1}, X_{1n}, \ldots, X_{n-1n}][X_{n-1n}^{-1}] \cong P_{t+1}(X)[x_{n-1n}^{-1}],$$

where x_{n-1n} denotes the residue class of X_{n-1n} in $P_{t+1}(X)$.

Proof. There exists an invertible matrix

$$C = \left(\begin{array}{ccc|cc} 1 & & 0 & 0 & 0 \\ & & & \vdots & \vdots \\ 0 & & 1 & 0 & 0 \\ \hline -X_{1n}X_{n-1n}^{-1} & \ldots & -X_{n-2n}X_{n-1n}^{-1} & 1 & 0 \\ X_{1n-1}X_{n-1n}^{-1} & \ldots & X_{n-2n-1}X_{n-1n}^{-1} & 0 & X_{n-1n}^{-1} \end{array}\right),$$

such that

$$C^tXC = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & Y & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \ldots & 0 & 0 & 1 \\ 0 & \ldots & 0 & -1 & 0 \end{array}\right).$$

The Laplace type expansion formula for pfaffians yields that the pfaffian of the matrix C^tXC equals the product of pfaffian of X with determinant of C. □

In order to distinguish between the pfaffians of X and Y we write $[\ ]_X$ and $[\ ]_Y$, respectively.

Lemma 4.1.10 (a) *One has* $[a_1, \ldots, a_s]_Y = [a_1, \ldots, a_s, n-1, n]X_{n-1n}^{-1}$.
(b) *Set* $R = K[X]$ *and* $S = K[X][X_{n-1n}^{-1}]$. *Then*

$$I_t(X)^{(k)} = I_t(X)^{(k)}S \cap R \quad \text{and} \quad I_{t+1}(X)^{(k)}S = I_t(Y)^{(k)},$$

for all $t \geq 1$.

Proof. (a) Use the invariance of pfaffians under elementary transformations.
(b) The first equation follows easily from $R \subset S \subset R_{I_t(X)}$. The second equation relies on the obvious fact that the extensions $R \subset S$ and $K[Y] \subset S$ commutes with the formation of symbolic powers. Then apply Lemma 4.1.9. □

It is a triviality that the symbolic powers and the ordinary powers coincide for $I_1(X)$. Thus we have $I_k(X) \subset I_1(X)^{(k)}$ and $I_k(X) \not\subset I_1(X)^{(k+1)}$ for all $1 \le k \le [n/2]$. Starting with this observation we can assert:

Proposition 4.1.11 *One has*

$$I_{t+k-1}(X) \subset I_t(X)^{(k)} \quad \textit{and} \quad I_{t+k-1}(X) \not\subset I_t(X)^{(k+1)},$$

for all $1 \le k \le [n/2] - t + 1$.

Proof. By induction on t and using Lemma 4.1.10. □

Denote by $I(t,k)$ the ideal generated by all the monomials ν with the property that $\gamma_t(\nu) \ge k$. Then $I(t,k) \subseteq I_t(X)^{(k)}$, with equality for $t = 1$. In order to prove the equality for all t by induction, via Lemma 4.1.10, we have to know that $I(t,k)S \cap R = I(t,k)$, or equivalently, that X_{n-1n} is not a zero-divisor modulo $I(t,k)$.

Lemma 4.1.12 *The standard monomials ν such that $\gamma_t(\nu) \ge k$ is a K-system of generators for $I(t,k)$. In particular, X_{n-1n} is not a zero-divisor modulo $I(t,k)$ if $t \ge 2$.*

Proof. It is enough to show that in a straightening relation

$$\pi_1\pi_2 = \sum \lambda_\nu \nu, \quad \lambda_i \in K, \quad \lambda_i \ne 0$$

we must have $\gamma_t(\nu) \ge \gamma_t(\pi_1) + \gamma_t(\pi_2)$ for all ν. By Theorem 1.0.14 we know that ν has at most two factors and ν and $\pi_1\pi_2$ have the same degree as polynomials in the entries of X.

The second statement is an easy consequence of the first: if ν is a standard monomial, then the product νX_{n-1n} is a standard monomial too, and $\gamma_t(\nu X_{n-1n}) = \gamma_t(\nu)$ for all $t \ge 2$. □

Now we can conclude:

Theorem 4.1.13 *The ideal $I_t(X)^{(k)}$ is the K-vector space generated by the standard monomials ν with $\gamma_t(\nu) \ge k$, and it contains all products of pfaffians ν' with $\gamma_t(\nu') \ge k$. One has*

$$I_t(X)^{(k)} = \sum_{\sigma=(s_1,\ldots,s_p),\ \gamma_t(\sigma)\ge k} I_{s_1}(X) \cdots I_{s_p}(X).$$

Similar to the generic case we can assert:

Corollary 4.1.14 *The symbolic powers of the ideal $I_t(X)$ coincide with the ordinary powers if and only if $t = 1$ or $2t \ge n - 1$.*

The primary decomposition of products of pfaffian ideals is very much alike to the primary decomposition of products of determinantal ideals.

Theorem 4.1.15 *Let* $\sigma = (t_1, \ldots, t_u)$ *be a non-increasing sequence of integers and suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(2t_i, n - 2t_i)$ *for* $i = 1, \ldots, u$. *Then*

$$I_{t_1}(X) \cdots I_{t_u}(X) = \bigcap_{i=1}^{t_1} I_i(X)^{(\gamma_i(\sigma))}$$

is a primary decomposition. In particular, $I_{t_1}(X) \cdots I_{t_u}(X)$ *is generated by the standard monomials* ν *with* $\gamma_i(\nu) \geq \gamma_i(\sigma)$ *for all* $i = 1, \ldots, t_1$, *and it contains all products of minors* ν' *satisfying these conditions.*

For ordinary powers of pfaffian ideals we have:

Corollary 4.1.16 *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(2t, n - 2t)$. *Then*

$$I_t(X)^k = \bigcap_{i=1}^{t} I_i(X)^{(k(t+1-i))}$$

is a primary decomposition. In particular, $I_t(X)^k$ *is generated by the standard monomials* ν *with* $\gamma_i(\nu) \geq k(t+1-i)$ *for all* $i = 1, \ldots, t$, *and it contains all products of minors* ν' *satisfying these conditions. Moreover this primary decomposition is irredundant if* $2t < n$ *and* $k \geq (w-1)/(w-t)$ *where* $w = [n/2]$.

The primary decomposition of $I_t(X)^k$ is irredundant for $k \gg 0$ if $2t < n$. Furthermore, independently of the characteristic the right side of the equality in Corollary 4.1.16 is the integral closure of $I_t(X)^k$; see De Negri [32].

In order to prove the Theorem 4.1.15 first observe that the inclusion $\subseteq$ is immediate: $\gamma_t(z) \geq \gamma_t(\sigma)$ for all $z \in I_{t_1}(X) \cdots I_{t_u}(X)$. For the converse inclusion we need the following lemma which is similar to [21, Lemma (10.10)]; see De Negri [32].

Lemma 4.1.17 *Let* r, s *be integers with* $0 \leq r \leq s-1$. *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(2r+2, n-(2r+2))$. *Then* $I_r(X)I_s(X) \subseteq I_{r+1}(X)I_{s-1}(X)$.

For a better understanding of Lemma 4.1.17, let us take a look to the first non-trivial case: $I_2(X)^2 = I_1^4 \cap (I_3 + I_2^2)$; indeed we assume $n \geq 6$. To get that $I_1^4 \cap (I_3 + I_2^2) \subseteq I_2(X)^2$, we must show that $I_1(X)I_3(X) \subseteq I_2(X)^2$, and this is the key point in the proof of the theorem.

Proof of Theorem 4.1.15. In view of Theorem 4.1.13 it is enough to show that a product $\nu = \pi_1 \cdots \pi_v$ is in $I_{t_1}(X) \cdots I_{t_u}(X)$ if $\gamma_t(\nu) \geq \gamma_t(\sigma)$ for all t. We use induction on u, the case $u = 1$ being trivial. If one of the pfaffians π_i has size t_1, we have through by induction. If not, we split ν into the product $\nu_1 = \pi_1 \cdots \pi_q$ of pfaffians of size $< t_1$, and $\nu_2 = \pi_{q+1} \cdots \pi_v$ of pfaffians of size $> t_1$ and continue as in [21, pg.127], applying Lemma 4.1.17. □

Finally, we assert a similar result to Corollary 4.1.6:

Corollary 4.1.18 *The ideal $I_t(X)^k$ has a K-basis consisting of all standard monomials ν with $\beta_k(\nu) \geq kt$.*

4.2 Powers of ideals of maximal minors (pfaffians)

(**G**) Let $X = (X_{ij})$ be an $m \times n$ generic matrix with $m \leq n$, $I = I_m(X)$ the ideal of $K[X]$ generated by all maximal minors of X, and τ a diagonal term order on $K[X]$. It is known that $I^{(i)} = I^i$ for all non-negative integers i; see Corollary 4.1.2. We shall prove that the set of all products $N_1 \cdots N_i$, where N_j are the maximal minors, is a Gröbner basis of the ith power of the ideal I for all i with respect to τ. Set $\mathcal{B} = \{\mu : \mu = \delta_1 \cdots \delta_p$ is a standard monomial, $\delta_1, \ldots, \delta_p$ are maximal minors, and $p \geq i\}$, and $\mathcal{S} = \{\xi_1 \cdots \xi_i : \xi_1, \ldots, \xi_i$ are maximal minors$\}$. The set $\mathcal{B}$ is a K-basis of I^i; see [21]. The main result of this part is the following:

Proposition 4.2.1 *The set $\mathcal{S}$ is a Gröbner basis of I^i with respect to τ, and therefore* $\operatorname{in}_\tau(I^i) = \operatorname{in}_\tau(I)^i$ *for all i.*

Proof. According to Lemma 3.4.4 the proof will be done if we show that for any $\mu \in \mathcal{B}$ there exists an $s \in \mathcal{S}$ such that $\operatorname{in}_\tau(s) | \operatorname{KRS}(\mu)$. Let $\mu \in \mathcal{B}, \mu = \delta_1 \cdots \delta_p$ a standard monomial with $p \geq i$, and

$$\operatorname{KRS}(\mu) = \begin{pmatrix} u_1 & & u_q \\ v_1 & & v_q \end{pmatrix},$$

where $q = mp$, and $v = (v_1, \ldots, v_q)$ a sequence of positive integers. The length of the longest increasing subsequence of v is m; see Theorem 3.2.1. Recall that $a_k(v)$ denotes the length of the longest subsequence of v which has no decreasing subsequences of length $k+1$. Any such sequence is obtained by taking the union of k increasing subsequences. In our case $a_i(v) = mi$, therefore there exists a subsequence of v that can be written as $\sigma_1 \uplus \cdots \uplus \sigma_i$, a disjoint union of increasing subsequences of v. Once again, $a_i(v) = mi$ implies that the length of the subsequences $\sigma_1, \ldots, \sigma_i$ is maximum, namely m, hence we get i increasing subsequences of v of length m. Set $\sigma_j = (v_{j1}, \ldots, v_{jm})$

with $v_{j1} < \cdots < v_{jm}$. Then the corresponding top row elements are ordered as follows $u_{j1} > \cdots > u_{jm}$. Now we consider the maximal minors $\xi_j = [v_{j1}, \ldots, v_{jm} | u_{jm}, \ldots, u_{j1}]$, $1 \leq j \leq i$, and their product $s = \xi_1 \cdots \xi_i$. It is obvious that $\mathrm{in}_\tau(s) | \mathrm{KRS}(\mu)$. □

We now present an application of Theorem 4.2.1 to the rings $K[X]/I_m(X)^i$. But first a general fact on regular sequences of indeterminates:

Lemma 4.2.2 *Let τ be a term order on a polynomial ring $T = K[X_1, \ldots, X_u]$ and $J \subset T$ an ideal. If the indeterminates $X_1, \ldots, X_v$ do not appear in the generators of $\mathrm{in}_\tau(J)$, then the residue classes of $X_1, \ldots, X_v$ form a regular sequence in T/J.*

Proof. It is clear that $\mathrm{in}_\tau(J + (X_1)) = \mathrm{in}_\tau(J) + (X_1)$. Therefore it is enough to prove the assertion for $v = 1$. If it would be false, we could choose an element $f \in T \setminus J$ such that $X_1 f \in J$ and suppose that f has the smallest initial term among all the elements with these properties. From $X_1 f \in J$ we get $X_1 \mathrm{in}_\tau(f) \in \mathrm{in}_\tau(J)$. It follows $\mathrm{in}_\tau(f) \in \mathrm{in}_\tau(J)$. Let $g \in J$ such that $\mathrm{in}_\tau(g) = \mathrm{in}_\tau(f)$, and set $f_1 = f - g$. We have $f_1 \in T \setminus J$, $X_1 f_1 \in J$, and $\mathrm{in}_\tau(f_1) < \mathrm{in}_\tau(f)$, a contradiction. □

From Theorem 4.2.1 and Lemma 4.2.2 we obtain:

Corollary 4.2.3 *The residue classes of the indeterminates X_{21}, X_{31}, X_{32}, $\ldots$, X_{m1}, $\ldots$, X_{mm-1}, X_{1n-m+2}, X_{1n-m+3}, $\ldots$, X_{1n}, X_{2n-m+3}, $\ldots$, X_{2n}, $\ldots$, X_{m-1n} form a regular sequence on the ring $K[X]/I^i$ for all nonnegative integers i.*

Proof. The above set of indeterminates is the complement of the set of indeterminates that appear in the generators of the initial ideal of I. □

Remark 4.2.4 The regular sequence given above is not a maximal one even for $i >> 0$. It has $m(m-1)$ elements, while $\min_{i \in \mathbb{N}} \mathrm{depth}\, K[X]/I^i = m^2 - 1$; see [46].

(S) Let $X = (X_{ij})$ be an $n \times n$ generic symmetric matrix, $I = I_{n-1}(X)$ the ideal of $K[X]$ generated by all maximal minors of X, and τ a diagonal term order on $K[X]$.

We shall prove that the set of all products $N_1 \cdots N_i$, where N_j are the maximal minors, is a Gröbner basis of the ith power of the ideal I for all i with respect to τ. Set $\mathcal{B} = \{M : M = \alpha_1 \cdots \alpha_p$ is a standard monomial, $\alpha_1, \ldots, \alpha_p$ are maximal minors, and $p \geq i\}$, and $\mathcal{S} = \{\xi_1 \cdots \xi_i : \xi_1, \ldots, \xi_i$ are maximal minors$\}$. The set $\mathcal{B}$ is a K-basis of I^i; see [1]. The main result of this part is the following

Proposition 4.2.5 *The set $\mathcal{S}$ is a Gröbner basis of I^i with respect to τ, and therefore* $\mathrm{in}_\tau(I^i) = \mathrm{in}_\tau(I)^i$ *for all i.*

Proof. In order to use Lemma 3.4.4 we have to show that for any $M \in \mathcal{B}$ there exists an $s \in \mathcal{S}$ such that $\mathrm{in}_\tau(s)|\mathrm{KRS}(M)$. Let $M \in \mathcal{B}, M = \alpha_1 \cdots \alpha_p$ a standard monomial with $p \geq i$, and

$$\mathrm{KRS}(M) = \begin{pmatrix} u_1 & \cdots & u_q \\ v_1 & & v_q \end{pmatrix},$$

where $q = (n-1)p$, and $v = v_1, \ldots, v_q$ a sequence of positive integers. The length of the longest increasing subsequence of v is $n-1$; see Theorem 3.2.1. Therefore we must have $a_i(v) = (n-1)i$, therefore there exists a subsequence of v that can be written as $\sigma_1 \uplus \cdots \uplus \sigma_i$, a disjoint union of increasing subsequences of v. Since $a_i(v) = (n-1)i$ we get that the length of the subsequences $\sigma_1, \ldots, \sigma_i$ is $n-1$, hence we get i strictly increasing subsequences of v of length $n-1$. Set $\sigma_j = (v_{j1}, \ldots, v_{jn-1})$ with $v_{j1} < \cdots < v_{jn-1}$. Then the corresponding top row elements are ordered as follow $u_{j1} > \cdots > u_{jn-1}$. Now we consider the maximal minors $\xi_j = [v_{j1}, \ldots, v_{jn-1}|u_{jn-1}, \ldots, u_{j1}]$, $1 \leq j \leq i$, and their product $s = \xi_1 \cdots \xi_i$. It is obvious that $\mathrm{in}_\tau(s)|\mathrm{KRS}(M)$. □

(A) Let $n = 2r+1$ be an odd number, X an $n \times n$ generic alternating matrix and $I = I_{n-1}(X)$ the ideal of $K[X]$ generated by all maximal pfaffians of X. We shall prove that the set of all products $N_1 \cdots N_i$, where N_j are the maximal pfaffians, $1 \leq j \leq n$, is a Gröbner basis of the ith power of the ideal I, for all i (with respect to the monomial term order τ introduced in chapter 3).

In this frame we have that $I^{(i)} = I^i$ for all non-negative integers i; see De Negri [32]. Set $\mathcal{B} = \{\nu : \nu = \pi_1 \cdots \pi_p$ is a standard monomial, $\pi_1, \ldots, \pi_p$ are maximal pfaffians, and $p \geq i\}$, and $\mathcal{S} = \{\xi_1 \cdots \xi_i : \xi_1, \ldots, \xi_i$ are maximal pfaffians$\}$. It is known that $\mathcal{B}$ is a K-basis of I^i; see Theorem 4.1.13. The main result of this part is the following:

Theorem 4.2.6 *The set $\mathcal{S}$ is a Gröbner basis of I^i with respect to τ, and therefore* $\mathrm{in}_\tau(I^i) = \mathrm{in}_\tau(I)^i$ *for all i.*

Proof. According to Lemma 3.4.4 the proof will be done if we show that for any $\mu \in \mathcal{B}$ there exists an $s \in \mathcal{S}$ such that $\mathrm{in}_\tau(s)|KRS(\mu)$. Let $\mu \in \mathcal{B}, \mu = \delta_1 \cdots \delta_p$ a standard monomial with $p \geq i$, and

$$KRS(\mu) = \begin{pmatrix} u_1 & \cdots & u_{2q} \\ v_1 & & v_{2q} \end{pmatrix},$$

where $2q = (n-1)p$, and $v = v_1, \ldots, v_{2q}$ a sequence of positive integers. The length of the longest strictly decreasing subsequence of v is $n-1$; see

Theorem 3.2.1. Let $d_k(v)$ denote the length of the longest subsequence of v which has no increasing subsequences of length $k+1$. Any such sequence is obtained by taking the union of k decreasing subsequences. In our case $d_i(v) = (n-1)i$, therefore there exists a subsequence of v that can be written as $\sigma_1 \uplus \cdots \uplus \sigma_i$, a disjoint union of strictly decreasing subsequences of v. But $d_i(v) = (n-1)i$ implies that the length of the subsequences $\sigma_1, \ldots, \sigma_i$ is maximum, namely $n-1$, hence we have got i strictly decreasing subsequences of v of length $n-1$. Set $\sigma_j = (v_{j1}, \ldots, v_{jn-1})$ with $v_{j1} > \cdots > v_{jn-1}$. Then the corresponding top row elements are ordered as follow $u_{j1} < \cdots < u_{jn-1}$. Now we consider the pfaffians $\xi_j = [u_{j1}, \ldots, u_{jr}, v_{jr}, \ldots, v_{j1}]$, $1 \le j \le i$, and their product $s = \xi_1 \cdots \xi_i$. It is obvious that $s \in \mathcal{S}$ and $\mathrm{in}_\tau(s) | \mathrm{KRS}(\mu)$. □

From Theorem 4.2.6 and Lemma 4.2.2 we get:

Corollary 4.2.7 *The residue classes of the indeterminates* $X_{12}, \ldots, X_{1n-2}, X_{23}, \ldots, X_{2n-3}, X_{34}, \ldots, X_{3n-4}, X_{3n}, X_{45}, \ldots, X_{4n-5}, X_{4n-1}, X_{4n}, \ldots, X_{rr+4}, \ldots, X_{rn}, X_{r+1r+3}, \ldots, X_{r+1n}, X_{r+2r+3}, \ldots, X_{r+2n}, \ldots, X_{n-1n}$ *form a regular sequence on the ring* $K[X]/I^i$ *for all nonnegative integers* i.

Proof. The above set of indeterminates is the complement of the set of indeterminates that appear in the generators of the initial ideal of I. □

Remark 4.2.8 The regular sequence given above is not a maximal one even for $i >> 0$. It has $(n-1)(r-1)$ elements, while $\min_{i\in\mathbb{N}} \mathrm{depth}\, K[X]/I^i = n(r-1)$; see [46].

4.3 Gröbner bases of powers of determinantal ideals

(G) For ordinary monomials M of $K[X]$ we define

$$\hat{\gamma}_t(M) = \sup\{\gamma_t(\mu) : \mu \text{ is a product of minors of } X \text{ with } \mathrm{in}_\tau(\mu) = M\},$$

$$\hat{\beta}_t(M) = \sup\{\beta_t(\mu) : \mu \text{ is a product of minors of } X \text{ with } \mathrm{in}_\tau(\mu) = M\},$$

where $\mathrm{in}_\tau(f)$ denotes the initial term of a polynomial f with respect to a diagonal term order τ. We will show that the functions $\hat{\gamma}_t$, respectively $\hat{\beta}_t$ describe the initial ideals of the symbolic, respectively ordinary powers of $I_t(X)$. We warn the reader that in general there does not exists a product of minors μ with $\mathrm{in}_\tau(\mu) = M$ and $\hat{\gamma}_t(M) = \gamma_t(\mu)$ for all t. For instance, take the monomial $M = X_{11}X_{13}X_{22}X_{34}X_{43}X_{45}$; see Example 4.3.7.

Schensted's theorem 3.2.1 can now be expressed in terms of the functions γ_t and $\hat{\gamma}_t$ as follows:

Corollary 4.3.1 *For a standard monomial μ one has*

$$\gamma_t(\mu) \neq 0 \text{ if and only if } \hat{\gamma}_t(\mathrm{KRS}(\mu)) \neq 0.$$

Proof. Write $\mu = \delta_1 \cdots \delta_u$ with δ_i an r_i-minor, $i = 1, \ldots, u$, $\mathrm{KRS}(\mu) = X_{u_1 v_1} \cdots X_{u_r v_r}$, and $v = v_1, \ldots, v_r$. Then from Theorem 3.2.1 $\gamma_t(\mu) = 0$ if and only if $r_1 < t$. On the other hand, $\hat{\gamma}_t(\mathrm{KRS}(\mu)) = 0$ if and only if $\gamma_t(\nu) = 0$ for all ν a product of minors with $\mathrm{in}_\tau(\nu) = M$, and this is equivalent with $r_1 = a_1(v) < t$. □

This suggests the following:

Theorem 4.3.2 *Let μ be a standard monomial. Then*

$$\gamma_t(\mu) = \hat{\gamma}_t(\mathrm{KRS}(\mu)) \text{ for all } t \geq 0.$$

It turns out that the Theorem 4.3.2 can be reduced to a theorem on the decomposition of sequences of integers into increasing subsequences. Let $v = v_1, \ldots, v_r$ be a sequence of integers. A decomposition g of v into increasing subsequences, an *inc-decomposition* for short, is said to have shape $\sigma = (s_1, \ldots, s_u)$ if its ith subsequence has length s_i. We set

$$\gamma_t(g) = \gamma_t(\sigma) \text{ and } \hat{\gamma}_t(v) = \sup\{\gamma_t(g) : g \text{ is an inc} - \text{decomposition of } v\},$$

$$\beta_t(g) = \beta_t(\sigma) \text{ and } \hat{\beta}_t(v) = \sup\{\beta_t(g) : g \text{ is an inc} - \text{decomposition of } v\}.$$

Inc-decompositions are crucial for our investigation since they describe realization of a monomial as an initial monomial of a product of minors. Consider $M = \mathrm{KRS}(\mu) = X_{u_1 v_1} \cdots X_{u_r v_r}$ with $u_1 \leq \cdots \leq u_r$ and $v_i \geq v_{i+1}$ whenever $u_i = u_{i+1}$. A representation of M as $\mathrm{in}_\tau(\nu)$ with ν a product of minors of shape $(s_1, \ldots, s_u)$ is equivalent to an inc-decomposition of the sequence $v = (v_1, \ldots, v_r)$ of shape $(s_1, \ldots, s_u)$. Consequently $\hat{\gamma}_t(M) = \hat{\gamma}_t(v)$, respectively $\hat{\beta}_t(M) = \hat{\beta}_t(v)$.

For any sequence of integers $v = (v_1, \ldots, v_r)$, let $P = \mathrm{INSERT}(v)$ be the standard tableau determined by v, and $\sigma = (s_1, \ldots, s_u)$ its shape. Then the equality $\hat{\beta}_t(v) = \beta_t(\sigma)$ is just an interpretation of Greene's theorem in terms of β-functions. We also have a similar statement for γ-functions.

Theorem 4.3.3 *Let $v = (v_1, \ldots, v_r)$ be a sequence of integers. Let $P = \mathrm{INSERT}(v)$ be the standard tableau determined by v, and $\sigma = (s_1, \ldots, s_u)$ its shape. Then $\gamma_t(\sigma) = \hat{\gamma}_t(v)$.*

Proof. Note that $\gamma_t(\sigma)$ represent the number of elements in the columns of P of index greater or equal than t. But this is nothing but the sum of the lengths of the columns of P of index greater or equal than t. Now we can show that $\gamma_t(\sigma) \geq w$ if and only if $\beta_k(\sigma) \geq (t-1)k + w$ for some k, $1 \leq k \leq t$. □

In spite of the Theorem 4.3.3, in general we cannot get an inc-decomposition of a sequence v with the same shape as INSERT(v). For the sequence $v = (4,1,2,5,6,3)$ in Example 3.1.6 the shape of INSERT(v) is $(4,2)$, but v has no inc-decomposition of shape $(4,2)$. However the shapes $(4,1,1)$ and $(3,3)$ occur, and this is sufficient for the invariance of the functions γ_t.

Now we are ready to describe Gröbner bases and/or initial ideals of (symbolic) powers, and more generally products of determinantal ideals.

Theorem 4.3.4 *The initial ideal* $\operatorname{in}_\tau(I_t^{(k)})$ *is the* K*-vector space generated by the ordinary monomials* M *with* $\hat{\gamma}_t(M) \geq k$. *In particular, a Gröbner basis of* $I_t(X)^{(k)}$ *is given by the set of bitableaux* Σ *with* $\gamma_t(\Sigma) = k$ *and no factor of size less than* t.

Proof. Once again we shall use Lemma 3.4.4. Set $\mathcal{S}$ be the set of the products of minors μ with $\gamma_t(\mu) \geq k$. By virtue of Theorem 4.1.1 we know that the set $\mathcal{B}$ of bitableaux Σ with $\gamma_t(\Sigma) \geq k$ is a K-basis of $I_t(X)^{(k)}$, and moreover $\mathcal{S} \subseteq I_t(X)^{(k)}$. Let $\Sigma \in \mathcal{B}$. Then by Theorem 4.3.2 we get that there exists $s \in \mathcal{S}$ with $\operatorname{in}_\tau(s) | \operatorname{KRS}(\Sigma)$, and thus the set $\mathcal{S}$ is a Gröbner basis of $I_t(X)^{(k)}$.

It remains to show that the initial term of any product of minors μ with $\gamma_t(\mu) \geq k$ is divisible by the initial term of a product of minors μ' with no factor of size less than t and with $\gamma_t(\mu') = k$. If the monomial μ has factors of size less than t, we simply get them rid. On the other side, if $\gamma_t(\mu) > k$ we then cancel $\gamma_t(\mu) - k$ places in the bitableau with the corresponding entries and thus get μ'. □

In order to describe the initial ideal of a product of determinantal ideals we need the following:

Lemma 4.3.5 *Let* I *and* J *be homogeneous ideals in* $K[X]$ *such that* $\operatorname{in}_\tau(I) = \operatorname{KRS}(I)$ *and* $\operatorname{in}_\tau(J) = \operatorname{KRS}(J)$. *Then* $\operatorname{in}_\tau(I) + \operatorname{in}_\tau(J) = \operatorname{in}_\tau(I+J) = \operatorname{KRS}(I+J)$ *and* $\operatorname{in}_\tau(I) \cap \operatorname{in}_\tau(J) = \operatorname{in}_\tau(I \cap J) = \operatorname{KRS}(I \cap J)$.

Proof. One has

$$\operatorname{KRS}(I+J) = \operatorname{KRS}(I) + \operatorname{KRS}(J) = \operatorname{in}_\tau(I) + \operatorname{in}_\tau(J) \subseteq \operatorname{in}_\tau(I+J),$$
$$\operatorname{KRS}(I \cap J) = \operatorname{KRS}(I) \cap \operatorname{KRS}(J) = \operatorname{in}_\tau(I) \cap \operatorname{in}_\tau(J) \supseteq \operatorname{in}_\tau(I \cap J),$$

and apply the Hilbert function argument. □

Application of Lemma 4.3.5 and Theorem 4.1.3 yields

Theorem 4.3.6 *Let $\sigma = (t_1, \dots, t_u)$ be a non-increasing sequence of integers. Set $g_i = \gamma_i(\sigma)$ and suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t_i, m - t_i, n - t_i)$ *for $i = 1, \dots, u$. Then*

$$\operatorname{in}_\tau(I_{t_1}(X)\cdots I_{t_u}(X)) = \bigcap_{i=1}^{t_1} \operatorname{in}_\tau(I_i(X)^{(g_i)}).$$

In particular, $\operatorname{in}_\tau(I_{t_1}(X)\cdots I_{t_u}(X))$ is generated, as a K-vector space, by the monomials M with $\hat\gamma_i(M) \geq \gamma_i(\sigma)$ for all $i = 1, \dots, t_1$.

As Bruns and Conca [16] remarked, the Theorem 4.3.6 is satisfactory if we want to determine the initial ideal of the product $I_{t_1}(X)\cdots I_{t_u}(X)$. However, it does not tell us how to find a Gröbner basis. The natural guess that a Gröbner basis of $I_{t_1}(X)\cdots I_{t_u}(X)$ is given by the products of minors, standard or not, which are in $I_{t_1}(X)\cdots I_{t_u}(X)$, proves to be wrong.

Example 4.3.7 Suppose that $m, n \geq 4$, $\operatorname{char} K = 0$ or > 3, and consider the ideal $I_2(X)I_4(X)$. This ideal has the following primary decomposition:

$$I_2(X)I_4(X) = I_1(X)^{(6)} \cap I_2(X)^{(4)} \cap I_3(X)^{(2)} \cap I_4(X).$$

The monomial $M = X_{11}X_{13}X_{22}X_{34}X_{43}X_{45}$ has $\hat\gamma_1(M) = 6$, $\hat\gamma_2(M) = 4$, $\hat\gamma_3(M) = 2$, $\hat\gamma_4(M) = 1$, hence $M \in \operatorname{in}_\tau(I_2(X)I_4(X))$. The products of minors of degree 6 in $I_2(X)I_4(X)$ have the shapes (6), or (5,1), or (4,2). The only initial term of a 4-monomial that divides M is $X_{11}X_{22}X_{34}X_{45}$, but the remaining factor $X_{13}X_{43}$ is not the initial term of a 2-monomial, and does not belong to $I_2(X)$. We can now deduce that M is not the initial term of a product of minors from $I_2(X)I_4(X)$.

Nevertheless, if we restrict our attention to powers of determinantal ideals, we get an optimal result.

Theorem 4.3.8 *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t, m - t, n - t)$. *Then the initial ideal $\operatorname{in}_\tau(I_t^k)$ is the K-vector space generated by the monomials M with $\hat\beta_i(M) \geq kt$. In particular, a Gröbner basis of $I_t(X)^k$ is given by the products of minors μ such that μ has at most k factors, $\beta_k(\mu) = kt$, and* $\deg(\mu) = kt$. *Therefore $I_t(X)^k$ has a minimal system of generators which is a Gröbner basis.*

Proof. Let $\mathcal{S}$ be the set of the products of minors μ with $\beta_k(\mu) \geq kt$. By virtue of Corollary 4.1.6 we know that the set $\mathcal{B}$ of standard bitableaux Σ with $\beta_k(\Sigma) \geq kt$ is a K-basis of $I_t(X)^k$, and moreover $\mathcal{S} \subseteq I_t(X)^k$. Greene's theorem 3.2.2 implies that for any standard bitableau $\Sigma \in \mathcal{B}$ there exists $\mu \in \mathcal{S}$ with $\operatorname{in}_\tau(\mu) | \operatorname{KRS}(\Sigma)$, and thus the set $\mathcal{S}$ is a Gröbner basis of $I_t(X)^k$.

It remains to show that the initial term of any product of minors μ with $\beta_t(\mu) \geq kt$ is divisible by the initial term of a product of minors μ' with at most k factors, $\deg(\mu') = kt$ and $\beta_t(\mu') = kt$. If the monomial μ has more than k factors, we skip some of the shortest minors. If $\beta_t(\mu) > kt$ we then cancel $\beta_t(\mu) - kt$ places from the first k rows of the corresponding bitableau. At last, if $\deg(\mu) > kt$, we then cancel $\deg(\mu) - kt$ places from the last rows. $\square$

(**A**) Despite its "not so nice" behaviour in the case of pfaffians, KRS behaves well with respect to the shapes. In our case it means that the corresponding γ- functions are still invariant under KRS, and we can prove a combinatorial result similar to Theorem 4.3.2.

Due to the switch we made in the definition of standard tableaux in this case, we must define a suitable analogous to the β-functions. For a shape $\sigma = (s_1, \ldots, s_p)$ we consider

$$\hat{\beta}_t^*(\sigma) = \beta_t(\sigma^*)$$

where σ^* is the dual shape of σ, and we extend this to products of pfaffians ν by setting:

$$\beta_t^*(\nu) = \beta_t^*(\sigma)$$

where σ is the shape of ν. For ordinary monomials M of $K[X]$ we define

$$\hat{\gamma}_t^*(M) = \sup\{\gamma_t(\nu) : \nu \text{ is a product of pfaffians of } X \text{ with } \mathrm{in}_\tau(\nu) = M\},$$

$$\hat{\beta}_t^*(M) = \sup\{\beta_t^*(\nu) : \nu \text{ is a product of pfaffians of } X \text{ with } \mathrm{in}_\tau(\nu) = M\},$$

where $\mathrm{in}_\tau(f)$ denotes the initial term of a polynomial f with respect to the term order τ introduced in Chapter 3.

Now we can state the following:

Theorem 4.3.9 *Let ν be a standard monomial. Then*

$$\gamma_t(\nu) = \hat{\gamma}_t(\mathrm{KRS}\,(\nu)) \text{ for all } t \geq 0.$$

This theorem, though similar to the Theorem 4.3.2, has a quite different proof. As in the previous part (G), the Theorem 4.3.9 can be reduced to a theorem on the decomposition of sequences of integers into decreasing subsequences. Let $v = (v_1, \ldots, v_r)$ be a sequence of integers. A decomposition g of v into decreasing subsequences, an *dec-decomposition* for short, is said to have shape $\sigma = (s_1, \ldots, s_u)$ if its ith subsequence has length s_i. We set

$$\gamma_t(g) = \gamma_t(\sigma) \text{ and } \hat{\gamma}_t(v) = \sup\{\gamma_t(g) : g \text{ is an dec} - \text{decomposition of } v\},$$

$$\beta_t^*(g) = \beta_t^*(\sigma) \text{ and } \hat{\beta}_t^*(v) = \sup\{\beta_t^*(g) : g \text{ is an dec} - \text{decomposition of } v\}.$$

Consider $M = \mathrm{KRS}(\nu) = X_{a_1b_1}\cdots X_{a_sb_s}$ with $a_i < b_i$ for all $i = 1,\dots,s$, $a_1 \le \cdots \le a_s$ and $b_i \le b_{i+1}$ whenever $a_i = a_{i+1}$. Then there exists ν' a product of pfaffians of shape $(s_1,\dots,s_r)$ with $\mathrm{in}_\tau(\nu') = M$ if and only if the sequence $b = (b_1,\dots,b_s)$ admits a dec-decomposition into decreasing subsequences of shape $(s_1,\dots,s_r)$. Therefore $\hat\gamma_t(M) = \hat\gamma_t(b)$ and $\hat\beta_t^*(M) = \hat\beta_t^*(b)$.

Analogous to inc-decompositions, dec-decompositions describe realization of a pfaffian as an initial monomial of a product of pfaffians. Consider $M = \mathrm{KRS}(\mu) = X_{u_1v_1}\cdots X_{u_rv_r}$ with $u_1 \le \cdots \le u_r$ and $v_i \ge v_{i+1}$ whenever $u_i = u_{i+1}$. A representation of M as $\mathrm{in}_\tau(\nu)$ with ν a product of pfaffians of shape $(s_1,\dots,s_u)$ is equivalent to a dec-decomposition of the sequence $v = (v_1,\dots,v_r)$ of shape $(s_1,\dots,s_u)$. Consequently $\hat\gamma_t(M) = \hat\gamma_t(v)$, respectively $\hat\beta_t^*(M) = \hat\beta_t^*(v)$.

For any sequence of integers $v = (v_1,\dots,v_r)$, let $P = \mathrm{INSERT}(v)$ be the standard tableau determined by v, and $\sigma = (s_1,\dots,s_u)$ its shape. Then the equality $\hat\beta_t^*(v) = \beta_t^*(\sigma)$ is just an interpretation of Greene's theorem in terms of β^*-functions.

Quite similar to Theorem 4.3.3 we can prove the following:

Theorem 4.3.10 *Let $b = (b_1,\dots,b_s)$ be a sequence of integers. Let $P = \mathrm{INSERT}(b)$ be the standard tableau determined by b, and $\sigma = (s_1,\dots,s_u)$ its shape. Then $\gamma_t(\sigma) = \hat\gamma_t(b)$.*

It is now obvious that the Theorem 4.3.9 follows immediately from the Theorem 3.3.11.

Now we describe Gröbner bases for the initial ideals of (symbolic) powers.

Theorem 4.3.11 *The initial ideal $\mathrm{in}_\tau(I_t^{(k)})$ is the K-vector space generated by the ordinary monomials M with $\hat\gamma_t(M) \ge k$. In particular, a Gröbner basis of $I_t(X)^{(k)}$ is given by the set of tableaux P with $\gamma_t(P) = k$ and no factor of size less than t.*

Proof. Just mimic the proof of Theorem 4.3.4. □

For products of pfaffian ideals we can describe their initial ideals as follows:

Theorem 4.3.12 *Let $\sigma = (t_1,\dots,t_u)$ be a non-increasing sequence of integers. Set $g_i = \gamma_i(\sigma)$ and suppose that* $\mathrm{char}\,K = 0$ *or* $\mathrm{char}\,K > \min(2t_i, n-2t_i)$ *for $i = 1,\dots,u$. Then*

$$\mathrm{in}_\tau(I_{t_1}(X)\cdots I_{t_u}(X)) = \bigcap_{i=1}^{t_1} \mathrm{in}_\tau(I_i(X)^{(g_i)}).$$

In particular, $\mathrm{in}_\tau(I_{t_1}(X)\cdots I_{t_u}(X))$ is generated, as a K-vector space, by the monomials M with $\hat\gamma_i(M) \ge \gamma_i(\sigma)$ for all $i = 1,\dots,t_1$.

Similarly to the generic case, we can say that the natural guess that a Gröbner basis of $I_{t_1}(X)\cdots I_{t_u}(X)$ is given by the products of pfaffians, standard or not, which are in $I_{t_1}(X)\cdots I_{t_u}(X)$, proves to be wrong.

Example 4.3.13 Suppose that $n=9$, $\operatorname{char} K = 0$ or > 3, and consider the ideal $I_2(X)I_4(X)$. This ideal has the following primary decomposition:

$$I_2(X)I_4(X) = I_1(X)^{(6)} \cap I_2(X)^{(4)} \cap I_3(X)^{(2)} \cap I_4(X).$$

The monomial $M = X_{19}X_{28}X_{67}X_{17}X_{36}X_{45}$ has $\hat{\gamma}_1(M) = 6$, $\hat{\gamma}_2(M) = 4$, $\hat{\gamma}_3(M) = 2$, $\hat{\gamma}_4(M) = 1$, hence $M \in \operatorname{in}_\tau(I_2(X)I_4(X))$. The products of pfaffians of degree 6 in $I_2(X)I_4(X)$ have the shapes $(4,2)$, or $(4,1,1)$, or $(3,3)$, or $(3,2,1)$, or $(2,2,2)$. The only initial term of a 8-pfaffian that divides M is $X_{19}X_{28}X_{36}X_{45}$, but the remaining factor $X_{67}X_{17}$ is not the initial term of a 4-pfaffian, and does not belong to $I_2(X)$. Since the cases $(3,2,1)$ and $(2,2,2)$ are obvious, it remains the case of shape $(3,3)$. In this case the only possibility for M is to be equal to the product of the initial terms of [126789] and [134567]. But the product of this two pfaffians does not belong to $I_2(X)I_4(X)$. We can now deduce that M is not the initial term of a product of pfaffians from $I_2(X)I_4(X)$.

However, if we restrict to the powers of pfaffian ideals, we can provide a Gröbner basis.

Theorem 4.3.14 *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(2t, n-2t)$. *Then the initial ideal* $\operatorname{in}_\tau(I_t^k)$ *is the K-vector space generated by the monomials M with* $\hat{\beta}_i^*(M) \geq kt$. *In particular, a Gröbner basis of* $I_t(X)^k$ *is given by the products of pfaffians ν such that ν has at most k factors,* $\beta_k^*(\nu) = kt$, *and* $\deg(\nu) = kt$. *Therefore* $I_t(X)^k$ *has a minimal system of generators which is a Gröbner basis.*

Chapter 5

Simplicial Complexes Associated to Determinantal Ideals

The goal of this chapter is to show how one can use the theory of Gröbner bases and the simplicial complexes in proving that the determinantal rings are Cohen-Macaulay and in the computation of the multiplicity (resp. a-invariant) of determinantal ideals. Herzog and Trung [45], respectively Conca [22] found a formula for the multiplicity of $R_t(X)$ and $P_t(X)$, respectively $S_t(X)$. The computation of the a-invariant was done by Bruns and Herzog [18] for $R_t(X)$ and $P_t(X)$, and by Conca [23] for $S_t(X)$.

It turns out that all these results can be proved by Gröbner deformation, that is, by the study of the rings $K[X]/\mathrm{in}_\tau(I(X,\delta))$. By Theorem 3.4.5 we know that $\mathrm{in}_\tau(I(X,\delta))$ is a square-free monomial ideal. The main feature of such ideals is that their residue class ring is the Stanley-Reisner ring of some simplicial complex.

5.1 Simplicial complexes

Let us describe the approach by recalling the main properties and notions to be used in the sequel. It is well known (see, for instance, Bruns and Herzog [19]) that the Hilbert function of a homogeneous ideal I in a polynomial ring R coincides to the one of the ideal $\mathrm{in}_\tau(I)$ generated by the leading terms of the polynomials in I with respect to a monomial order τ. If $\mathrm{in}_\tau(I)$ is generated by square-free monomials, which is the case for determinantal ideals, one can associate with $\mathrm{in}_\tau(I)$ a simplicial complex Δ such that $R/\mathrm{in}_\tau(I)$ is the Stanley-Reisner ring associated to Δ. To enter the details, let us say that a *simplicial complex* on a set of *vertices* $V = \{1,\dots,n\}$ is a collection Δ of subsets F of V such that $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$, and such that $\{i\} \in \Delta$ for all $i = 1, \ldots, n$. Starting from here, we associate to any square-free monomial ideal I in a polynomial ring $R = K[X_1, \ldots, X_n]$ a simplicial complex

$$\Delta = \{F \subseteq \{1, \ldots, n\} : X_F \notin I\}$$

where $X_F = \prod_{i \in F} X_i$. Conversely, to any simplicial complex Δ we associate a square-free monomial ideal I by setting

$$I = (X_F : F \notin \Delta).$$

The *Stanley-Reisner ring* of Δ is the graded K-algebra $K[\Delta] = K[X]/I$. An element $F \in \Delta$ is called a *face*. If we denote by $|F|$ the cardinality of F, then $\dim F$, the *dimension* of F, is $|F| - 1$, and the dimension of Δ is maximum of $\dim F$ for all $F \in \Delta$. Set $d - 1 = \dim \Delta$. Then we denote by f_i the number of i-dimensional faces of Δ. The d-tuple $f(\Delta) = (f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ. The maximal elements of Δ under inclusion are called *facets*, and denote by $\mathbf{F}_\Delta$ the set of facets of Δ. The simplicial complex Δ is said to be *pure* if all its facets have the same dimension, i.e. $\dim F = \dim \Delta$ for all $F \in \mathbf{F}_\Delta$. As we will see soon, the study of the homological properties and the determination of the numerical invariants of $K[\Delta]$ reduces to the analyze of the combinatorial properties and invariants of Δ. Here is the first example involving the Krull dimension and the multiplicity of $K[\Delta]$:

Proposition 5.1.1 *We have that* $\dim K[\Delta] = \dim \Delta + 1$, *and the multiplicity* $e(K[\Delta])$ *equals* f_{d-1} *the number of facets of maximal dimension of* Δ.

Proof. See Bruns and Herzog [19, Chapter 5]. □

(We shall see that in the case of determinantal ideals the facets can be characterized as families of non-intersecting paths.)

The Hilbert series of the Stanley-Reisner rings can be expressed in terms of the f-vector of Δ:

Proposition 5.1.2 *For a simplicial complex Δ of dimension $d - 1 \geq 0$ with f-vector $(f_0, \ldots, f_{d-1})$, the Hilbert series of the associated Stanley-Reisner ring is given by*

$$H_{K[\Delta]}(z) = 1 + \sum_{i=0}^{d-1} \frac{f_i z^{i+1}}{(1-z)^{i+1}}.$$

A distinguished class of simplicial complexes which gives rise to Cohen-Macaulay rings is that of shellable simplicial complexes. A pure simplicial complex Δ is called *shellable* if its facets can be given a total order $F_1, \ldots, F_m$ such that the following condition holds: for all i and j with $1 \leq j < i \leq m$ there exists $v \in F_i \setminus F_j$ and $k \in \{1, \ldots, i-1\}$ such that $F_i \setminus F_k = \{v\}$. A total order of the facets which satisfy this condition is called a *shelling* of Δ. The advantage of using shellable simplicial complexes is given by the fact that these are suitable for inductive arguments. For instance, the next result can be proved by induction on the number of facets of Δ:

Theorem 5.1.3 *Let Δ be a shellable simplicial complex. Then the Stanley-Reisner ring $K[\Delta]$ is Cohen-Macaulay.*

Recall from Bruns and Herzog [19, Corollary 4.1.8] that the Hilbert series of a homogeneous K-algebra R (homogeneous means that R is generated, as K-algebra, by elements of degree 1) of Krull dimension d has the form $H_R(z) = h(z)/(1-z)^d$ where $h(z) = \sum h_i z^i \in \mathbb{Z}[z]$ and $h(1) \neq 0$. By definition, the *a-invariant* of R is given by $a(R) = \deg h - d$. When $R = K[\Delta]$, the finite sequence of integers $h(\Delta) = (h_0, h_1, \ldots)$ is said to be the *h-vector* of Δ, while $a(R)$, denoted by $a(\Delta)$, is called the *a-invariant of* Δ.

For a shellable simplicial complex Δ with shelling $F_1, \ldots, F_m$ we set

$$c(F_i) = \{v \in F_i : \text{ there exists } k < i \text{ such that } F_i \setminus F_k = \{v\}\}$$

with $1 \leq i \leq m$. There is a combinatorial interpretation of the h-vector of a shellable simplicial complex due to McMullen and Walkup; see Bruns and Herzog [19, Corollary 5.1.14].

Proposition 5.1.4 *Let Δ be a shellable simplicial complex of dimension $d - 1 \geq 0$ with shelling $F_1, \ldots, F_m$. Then $h_0 = 1$ and $h_j = |\{i : |c(F_i)| = j\}|$.*

There is also a combinatorial interpretation of the a-invariant of a shellable simplicial complex; see Bruns and Herzog [18, Prop. 2.1]. In order to apply this to the weighted case we need to broad the setting. A simplicial complex Δ on a set $V = \{v_1, \ldots, v_n\}$ together with a map $\varphi : V \longrightarrow \mathbb{N}$ is called a *weighted simplicial complex*. The number $\varphi(v_i)$ is said to be the weight of v_i. The Stanley-Reisner ring $K[\Delta]$ is a positively graded K-algebra by setting $\deg X_i = a_i$ where $a_i = \varphi(v_i)$. The Hilbert series $H_{K[\Delta]}(z)$ of $K[\Delta]$ is of the form

$$H_{K[\Delta]}(z) = \frac{h(z)}{\prod_{i=1}^{n}(1 - z^{a_i})}$$

with $h(z) \in \mathbb{Z}[z]$ and $h(1) \neq 0$. The a-invariant of Δ can be written $a(\Delta) = \deg h - \sum_{i=1}^{n} a_i$.

Proposition 5.1.5 *Let Δ be a weighted simplicial complex whose weights are $\varphi(v_i) = a_i$, $i = 1, \ldots, n$. Suppose Δ is shellable of dimension $d - 1 \geq 0$ with shelling $F_1, \ldots, F_m$. Set $b_i = \sum_{v_j \in F_i \setminus c(F_i)} a_j$. Then $a(\Delta) = -\min\{b_1, \ldots, b_m\}$.*

Note that in the homogeneous case, that is $a_i = 1$ for all i, by Proposition 5.1.5 we get $a(\Delta) = -\min\{d - |c(F_1)|, \ldots, d - |c(F_m)|\}$.

5.2 Cohen-Macaulayness and the multiplicity of determinantal rings

(G) Let τ be a diagonal term order on $K[X]$, $\delta = [a_1, \dots, a_t \mid b_1, \dots, b_t] \in \Delta(X)$ and set $a_{t+1} = m+1$ and $b_{t+1} = n+1$. We know that the initial term of δ with respect to τ is $\text{in}_\tau(\delta) = X_{a_1 b_1} \cdots X_{a_t b_t}$. As we already noticed, the Hilbert series of $R(X, \delta)$ coincides with the Hilbert series of $R/\text{in}_\tau(I(X,\delta))$. The initial ideal $\text{in}_\tau(I(X,\delta))$ is a monomial ideal generated by square-free monomials of the form $X_{a'_1 b'_1} \cdots X_{a'_s b'_s}$ with $a'_i \geq a_i$, $b'_i \geq b_i$ for all $i = 1, \dots, s-1$ and $a'_s < a_s$ or $b'_s < b_s$, $s = 1, \dots, t+1$. In order to describe the corresponding simplicial complex, let us consider the following subset of the plane

$$V = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

It is a finite poset with the partial order

$$(i,j) \leq (i',j') \text{ if and only if } i \geq i' \text{ and } j \leq j'.$$

One observes that two pairs (i,j) and (i',j') are incomparable if $i < i'$, $j < j'$ or $i > i'$, $j > j'$. A subset of V is said to be a *chain* if any two of its elements are comparable, and it is an *antichain* whether it does not contain a pair of comparable elements. Therefore an antichain of V of length s (for short s-antichain) is a family of elements $(u_1, v_1), \dots, (u_s, v_s)$ with the property $u_1 < \cdots < u_s$, $v_1 < \cdots < v_s$. Observe that each generator of the initial ideal $\text{in}_\tau(I(X,\delta))$ of the form $X_{a'_1 b'_1} \cdots X_{a'_s b'_s}$ has a corresponding family of elements in $D_s = \{(i,j) : i < a_s \text{ or } j < b_s\}$, namely $(a'_1, b'_1), \dots, (a'_s, b'_s)$ with $a'_i \geq a_i$, $b'_i \geq b_i$, therefore an s-antichain. The corresponding simplicial complex of the initial ideal $\text{in}_\tau(I(X,\delta))$ is

$$\Delta_\delta = \{Z \subseteq V : Z \text{ does not contain } s\text{-antichains of the set } D_s \text{ for any } s = 1, \dots, t+1\},$$

and therefore $R/\text{in}_\tau(I(X,\delta))$ is the Stanley-Reisner ring of Δ_δ. Set $d - 1 = \dim \Delta_\delta$ and let f_i be the number of i-dimensional faces of Δ_δ, $i = 0, \dots, d-1$. In particular, the multiplicity of the determinantal ring $R(X, \delta)$ is f_{d-1}. We shall see that the number f_{d-1} of the faces of maximal dimension of Δ_δ can be interpreted as the number of certain families of non-intersecting paths of V.

By a *path* in V starting from a point P and ending to a point Q, $P \geq Q$, we mean a maximal chain between P and Q. It can be written as a sequence of points

$$P = (u_1, v_1), \dots, (u_s, v_s) = Q$$

with

$$(u_i - u_{i-1}, v_i - v_{i-1}) \text{ equals either } (1,0) \text{ or } (0,-1) \text{ for all } i = 2, \dots, s.$$

A point (u_i, v_i) of a given path is called a *right-turn* of the path if $1 < i < s$ and

$$(u_{i+1} - u_i, v_{i+1} - v_i) = (0, -1) \text{ and } (u_i - u_{i-1}, v_i - v_{i-1}) = (1, 0).$$

If $\mathcal{P} = P_1, \dots, P_s$ and $\mathcal{Q} = Q_1, \dots, Q_s$ are two sets of s points in V, then a *family of non-intersecting paths* from $\mathcal{P}$ to $\mathcal{Q}$ is a union of paths $\mathcal{C} = \cup_{i=1}^{s} \mathcal{C}_i$ from P_i to Q_i, $i = 1, \dots, s$, such that $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for all $i \neq j$. A point $C \in \mathcal{C}$ is said to be a right-turn of $\mathcal{C}$ if it is a right-turn of the path to which it belongs. It is an interesting fact (and easy to see) that the number of points of any path from two points $P = (u, v)$ and $Q = (u', v')$ depends only on P and Q. It is equal to $u' - u + v - v' + 1$ and is called the *length* of the path.

Set $P_i = (a_i, n)$, and $Q_i = (m, b_i)$ with $i = 1, \dots, t$. Our goal is to prove that the faces of maximal dimension of Δ_δ can be described as being families of non-intersecting paths from $\mathcal{P} = P_1, \dots, P_t$ to $\mathcal{Q} = Q_1, \dots, Q_t$. We first introduce some notation. For each element $x = (a, b) \in V$ we set

$$L_x = \{(i, j) \in V : i < a, j < b\}, \quad \overline{L}_x = \{(i, j) \in V : i \leq a, j \leq b\},$$

$$R_x = \{(i, j) \in V : i > a, j > b\}, \quad \overline{R}_x = \{(i, j) \in V : i \geq a, j \geq b\},$$

while for any subset $Z \subseteq V$ we set

$$L_Z = \cup_{x \in Z} L_x, \ \overline{L}_Z = \cup_{x \in Z} \overline{L}_x, \ R_Z = \cup_{x \in Z} R_x, \text{ respectively } \overline{R}_Z = \cup_{x \in Z} \overline{R}_x.$$

Further, for a subset Z of V we define

$$b(Z) = \{x \in Z : L_x \cap Z = \emptyset\}.$$

Note that $b(Z)$ is, in some sense, the left border of Z. It is easily seen that $b(Z)$ is always a chain of V. Now set

$$Z_1 = b(Z) \text{ and } Z_i = b\Big(Z \setminus \bigcup_{j=1}^{i-1} Z_j\Big) \text{ for } i > 1.$$

This is called the *light and shadow decomposition* of Z, where the light comes from the point $(1, 1)$. It is obvious that the family of the Z_i is uniquely determined by the following conditions:

(a) Z_i is a chain,

(b) $Z = \bigcup_{i \geq 1} Z_i$,

(c) $Z_i \subset R_{Z_{i-1}}$, i.e. Z_i lies strictly on the upper-right side of Z_{i-1} for all $i > 1$.

Set $V_s = V \setminus D_s$. We can give another description of the faces of Δ_δ in terms of the light and shadow decomposition:

Lemma 5.2.1 *We have* $\Delta_\delta = \{Z \subseteq V : Z_s \subseteq V_s$ *for all* $s = 1, \ldots, t+1\}$.

Proof. Suppose that $Z_s \nsubseteq V_s$ for some $s \in \{1, \ldots, t+1\}$. Then there exists an element $x_s = (u_s, v_s) \in Z_s$ such that $x_s \notin V_s$. In particular, we have $x_s \notin Z_{s-1}$, hence there exists an $x_{s-1} = (u_{s-1}, v_{s-1}) \in Z \setminus \cup_{i<s-1} Z_i$ such that $u_{s-1} < u_s$, $v_{s-1} < v_s$. Actually one can take $x_{s-1} \in Z_{s-1}$. Similarly we get a sequence of elements $x_i = (u_i, v_i) \in Z_i$, $i = t-2, \ldots, 1$, with the property that $u_{s-1} > \cdots > u_1$, $v_{s-1} > \cdots > v_1$. In this way we have found an s-antichain in $Z \cap D_s$, a contradiction.

Conversely, since $Z_s \subseteq V_s$ we have $Z_i \subseteq V_s$ for all $i \geq s$. It follows that $Z_i \cap D_s = \emptyset$ for all $i \geq s$. On the other side, $Z \cap D_s = \cup_{j<t}(Z_j \cap D_s)$ is a decomposition of $Z \cap D_s$ into less than s disjoint chains. Since the elements of an antichain of $Z \cap D_s$ must belong to different chains, it follow that $Z \cap D_s$ cannot contain any s-antichain, and we are done. □

Now we are ready to prove that the faces of maximal dimension of Δ_δ can be described as families of non-intersecting paths from $\mathcal{P} = P_1, \ldots, P_t$ to $\mathcal{Q} = Q_1, \ldots, Q_t$.

Theorem 5.2.2 *A subset* Z *of* V *is a face of* Δ_δ *of maximal dimension, i.e.* $\dim Z = \dim \Delta_\delta$ *if and only if* Z *is a family of non-intersecting paths from* $\mathcal{P}$ *to* $\mathcal{Q}$.

Proof. Let $Z \in \Delta_\delta$. By Lemma 5.2.1 we know that $Z_s \subseteq V_s$ for all $s = 1, \ldots, t+1$. In particular $Z_{t+1} = \emptyset$, hence $Z = \cup_{s=1}^t Z_s$. As Z_s is a subset of V_s, we get that the length Z_s is bounded above by the length of a path from P_s to Q_s, which is equal to $m + n - a_s - b_s + 1$. Summing up we get

$$|Z| = \sum_{i=1}^{t} |Z_i| \leq t(m+n+1) - \sum_{i=1}^{t}(a_i + b_i).$$

Let W_s be the (unique) path from P_s to Q_s passing through the point (a_s, b_s), $s = 1, \ldots, t$, and consider $W = \cup_{s=1}^t W_s$. Obviously $W \in \Delta_\delta$ and $|W| = t(m+n+1) - \sum_{i=1}^t (a_i + b_i)$. In particular, we get that

$$\dim \Delta_\delta = t(m+n+1) - \sum_{i=1}^{t}(a_i + b_i) - 1.$$

Since $\dim Z = \dim \Delta_\delta$ we must have $|Z_s| = m + n - a_s - b_s + 1$ for all s, and it happens only when Z_s is a path from P_s to Q_s.

Conversely, if Z is a family of non-intersecting paths from $\mathcal{P}$ to $\mathcal{Q}$, then these paths are the Z_i in the light and shadow decomposition of Z. Obviously $Z_i \subseteq D_i$ for all i, hence $Z \in \Delta_\delta$ and $\dim Z = \dim \Delta_\delta$. □

Remark 5.2.3 (a) From the preceding results it follows that any face of Δ_δ can be describe as a disjoint union of chains Z_s contained in V_s with $s = 1, \ldots, t$.

(b) Note that V_s is a distributive lattice with minimal element Q_s and maximal element P_s. We further get that $Z \in \Delta_\delta$ is a facet if and only if Z_s is a maximal chain of V_s for all $s = 1, \ldots, t$. It is clear that a maximal chain in V_s is a path from P_s to Q_s. Conversely, if Z is a facet of Δ_δ whose light and shadow decomposition is $Z = \cup_{i=1}^t Z_i$, let us suppose that Z_1 is not a maximal chain in V_1. Since Z_1 is not a maximal chain in V_1, it can be extended to a maximal one, let us say Z_1'. Replacing Z_1 by Z_1' we get another face Z' of Δ_δ such that Z is strictly contained in Z', a contradiction. (The argument can continues similarly for $Z_2, \ldots, Z_t$.) From this we thus get that the complex Δ_δ is pure.

(c) As a by-product of the proof of Theorem 5.2.2 and Proposition 5.1.1 we can regain the dimension formula of Proposition 1.0.9.

By Theorem 5.2.2 and purity of Δ_δ it follows that the multiplicity of $K[\Delta_\delta]$, and, of course, that of $R(X, \delta)$ is given by the number of families of non-intersecting paths from $\mathcal{P}$ to $\mathcal{Q}$. To compute this number we use the Gessel-Viennot [39] determinantal formula: for any two sequences of points $\mathcal{P} = P_1, \ldots, P_t$ and $\mathcal{Q} = Q_1, \ldots, Q_t$, the number $\mathrm{Paths}(\mathcal{P}, \mathcal{Q})$ of families of non-intersecting paths from $\mathcal{P}$ to $\mathcal{Q}$ is given by the formula

$$\mathrm{Paths}(\mathcal{P}, \mathcal{Q}) = \det\left(\mathrm{Paths}(P_i, Q_j)\right)_{i,j=1,\ldots,t}$$

where $\mathrm{Paths}(P_i, Q_j)$ denotes the number of paths from P_i to Q_j. In our case, $P_i = (a_i, n)$ and $Q_j = (m, b_j)$. It is easy to see that $\mathrm{Paths}(P_i, Q_j) = \binom{m+n-a_i-b_j}{m-a_i}$ and thus

Theorem 5.2.4 *We have*

$$e(R(X,\delta)) = \det\left(\binom{m+n-a_i-b_j}{m-a_i}\right)_{i,j=1,\ldots,t}.$$

For the classical determinantal rings $R_t(X)$ it yields the formula

$$e(R_t(X)) = \det\left(\binom{m+n-i-j}{m-i}\right)_{i,j=1,\ldots,t-1}$$

which can be simplified by using Vandermonde's determinant to the following:

Corollary 5.2.5 *We have*

$$e(R_t(X)) = \prod_{i=0}^{n-t} \frac{(m+i)!i!}{(t-1+i)!(m-t+1+i)!}. \tag{5.1}$$

Next we prove that Δ_δ is shellable. This is a special case of a theorem of Björner [12] on the higher order complexes of finite planar distributive lattices.

Theorem 5.2.6 *The simplicial complex Δ_δ is shellable.*

Proof. By Remark 5.2.3(b) we have that all facets of Δ_δ have the same dimension.

It remains to prove that there exists a shelling on the facets of Δ_δ. We start by giving a partial order on the set of paths connecting two fixed points $P, Q \in V$ with $P \leq Q$. Let Z_1 and Z_1' be two paths from P to Q. We say that $Z_1' \leq Z_1$ if Z_1 is on the upper-right side of Z_1', i.e. $Z_1 \subseteq \overline{R}_{Z_1'}$, and this is a partial order. For any two sequences of points $\mathcal{P} = P_1, \ldots, P_t$ and $\mathcal{Q} = Q_1, \ldots, Q_t$, and two families of non-intersecting paths $Z = \cup_{i=1}^t Z_i$ and $Z' = \cup_{i=1}^t Z_i'$ from $\mathcal{P}$ to $\mathcal{Q}$, we set $Z' \leq Z$ if and only if $Z_i' \leq Z_i$ for all i. Then extend this partial order arbitrarily to a total order. To prove that the resulting total order is a shelling on the the facets of Δ_δ, we take Z and Z' facets and suppose that $Z' < Z$. Then $Z_s' \not\subseteq \overline{R}_{Z_s}$ for some $s = 1, \ldots, t$, and we choose s as being the largest index with this property. In particular, we must have $Z_{s+1}' \subseteq \overline{R}_{Z_{s+1}}, \ldots, Z_t' \subseteq \overline{R}_{Z_t}$. Since $Z_s' \not\subseteq \overline{R}_{Z_s}$, there exists an element $y \in Z_s' \setminus \overline{R}_{Z_s}$. From all elements $x \in Z_s$ such that $y \in L_x$ we choose an $x = (a, b)$ with the property that $\overline{R}_x \cap Z_s = \{x\}$. Note that such an element always exists and it should be a right-turn of $Z_s \cap R_y$. Let $x_1 = (a-1, b-1)$. If $x_1 \notin Z_{s-1}$ (or $s = 1$), we consider $Z_i'' = Z_i$ for $i \neq s$ and $Z_s'' = (Z_s \setminus \{x\}) \cup \{x_1\}$. Otherwise, we need to regress more and let us consider the element $x_2 = (a-2, b-2)$. If $x_2 \notin Z_{s-2}$ (or $s = 2$), we set $Z_i'' = Z_i$ for $i \neq s-1, s$, $Z_s'' = (Z_s \setminus \{x\}) \cup \{x_1\}$, and $Z_{s-1}'' = (Z_{s-1} \setminus \{x_1\}) \cup \{x_2\}$. The general case follows in the same manner. Finally we get a facet Z'' of Δ_δ such that $Z'' < Z$, $Z \setminus Z'' = \{x\}$, and obviously $x \in Z \setminus Z''$. □

Remark 5.2.7 The proof of Theorem 5.2.6 shows that the facets of Δ_δ can be given a shelling such that $c(F)$ is the set of right-turns of F for each facet $F \in \Delta_\delta$.

One of the main consequences of Theorem 5.2.6 is the following:

Theorem 5.2.8 *The rings $K[\Delta_\delta]$ and $R(X, \delta)$ are Cohen-Macaulay.*

Proof. By Theorem 5.2.6 it follows that Δ_δ is shellable, hence $K[\Delta_\delta]$ is Cohen-Macaulay by Theorem 5.1.3. Thus we get that the ring $K[X]/\text{in}_\tau(I(X, \delta))$ is Cohen-Macaulay, and this is enough to see that $R(X, \delta)$ itself is Cohen-Macaulay; see Eisenbud [35]. □

(S) Let τ be a diagonal term order on $K[X]$, $\alpha = \{\alpha_1, \ldots, \alpha_t\} \in H$ and set $\alpha_{t+1} = n + 1$. The initial ideal $\text{in}_\tau(I(X, \alpha))$ is a monomial ideal generated by square-free monomials of the form $X_{a_1b_1}\cdots X_{a_sb_s}$ with $a_i \geq \alpha_i$ for all $i = 1,\dots,s-1$ and $a_s < \alpha_s$, $s = 1,\dots,t+1$. In order to describe the corresponding simplicial complex, let us consider $V = \{(i,j) : 1 \leq i,j \leq n\}$ and its subset $V' = \{(i,j) \in V : i \leq j\}$. Observe that these are finite posets with the partial order

$$(i,j) \leq (i',j') \text{ if and only if } i \geq i' \text{ and } j \leq j'.$$

Set $D_s = \{(i,j) : i < \alpha_s \text{ or } j < \alpha_s\}$ and note that each generator of the initial ideal $\mathrm{in}_\tau(I(X,\alpha))$ of the form $X_{a_1b_1}\cdots X_{a_sb_s}$ has a corresponding family of elements in $D'_s = D_s \cap V'$, namely $(a_1,b_1),\dots,(a_s,b_s)$ with $a_i \geq \alpha_i$, therefore an s-antichain in D'_s. We define

$$\Delta_\alpha = \{Z \subseteq V : Z \text{ does not contain } s\text{-antichains of the set } D_s \text{ for any } s = 1,\dots,t+1\},$$

and let Δ'_α be the restriction of Δ_α to V'. The corresponding simplicial complex of the initial ideal $\mathrm{in}_\tau(I(X,\alpha))$ is Δ'_α, and therefore $R/\mathrm{in}_\tau(I(X,\alpha))$ is the Stanley-Reisner ring of Δ'_α. Using the results of part (G) which describe the facets of Δ_α we will be able to describe the facets of Δ'_α. The first observation is that Δ_α is the simplicial complex Δ_δ defined in part (G) with $\delta = [\alpha_1,\dots,\alpha_t|\alpha_1,\dots,\alpha_t]$. Recall that a facet Z of Δ_δ is the disjoint union of $Z_1,\dots,Z_t$ where Z_s is a maximal chain of the distributive lattice $V \setminus D_s$. A maximal chain of $V \setminus D_s$ may be interpreted as a path from (n,α_s) to (α_s,n).

Lemma 5.2.9 *Let Z be a facet of Δ_α. Then $|Z \cap V'| = (n+1)t - \sum_{i=1}^t \alpha_i$.*

Proof. The rank of the poset $V \setminus D_s$ is clearly $2(n-\alpha_s)+1$. The elements of $V' \setminus D'_s$ are exactly the elements of $V \setminus D_s$ of rank greater or equal than $n+1-\alpha_s$. Thus any maximal chain of $V \setminus D_s$ must contain exactly $n+1-\alpha_s$ elements of $V' \setminus D'_s$. Now the result follows from the characterization of the facets of Δ_α; see Theorem 5.2.2. □

As an immediate consequence we get:

Proposition 5.2.10 *Let Z' be a face of Δ'_α. Then Z' is a facet of Δ'_α if and only if there exists a facet Z of Δ_α such that $Z' = Z \cap V'$.*

Proof. Let first Z' be be a facet of Δ'_α. Since Δ'_α is the restriction of Δ_α to V' there exists a facet Z of Δ_α such that $Z' \subseteq Z$. On the other side, $Z \cap V'$ is a face of Δ'_α which contains Z', and so $Z' = Z \cap V'$. Conversely, first note that Z' is contained in a facet W' of Δ'_α. As we already know $W' = W \cap V'$ with W a facet of Δ_α. By virtue of Lemma 5.2.9 the facets Z and W of Δ_α must have the same cardinality, and hence $Z = W$. □

Lemma 5.2.9 and Proposition 5.2.10 provide an immediate proof for Proposition 1.0.13(a). Let us recall it:

Corollary 5.2.11 *We have* $\dim R(X,\alpha) = (n+1)t - \sum_{i=1}^{t} \alpha_i$.

Now we describe the facets of Δ'_α in terms of disjoint union of paths. Let $P_s = (\alpha_s, n)$ and $Q_s = \{(\alpha_s,\alpha_s), (\alpha_s+1,\alpha_s+1), \ldots, (n,n)\}$, $s = 1,\ldots,t$.

Proposition 5.2.12 *The set Z' is a facet of Δ'_α if and only if Z' is the disjoint union of $Z'_1,\ldots,Z'_t$, where Z'_s is a path from P_s to one of the points of Q_s, $s = 1,\ldots,t$.*

Proof. By Proposition 5.2.10 we know that Z' is a facet of Δ'_α if and only if there exists a facet Z of Δ_α such that $Z' = Z \cap V'$. But Z is the union of disjoint paths $Z_1,\ldots,Z_t$, where Z_s is a path from P_s to (n,α_s). If we set $Z'_s = Z_s \cap V'$ then Z'_s is a maximal chain in $V' \setminus D'_s$. Now take into account that the set of minimal elements of $V' \setminus D'_s$ is exactly Q_s. □

The above description of the facets of Δ'_α and the purity of the simplicial complex Δ_α show that the simplicial complex Δ'_α itself is pure. It follows that the multiplicity of $K[\Delta'_\alpha]$, and, of course, that of $R(X,\alpha)$ is also a matter of paths-counting, the only difference being that the ending points are not fixed now.

Theorem 5.2.13 *We have*

$$e(R(X,\alpha)) = \sum_{1\le k_1<\cdots<k_t\le n,\ \alpha_i\le k_i} \det\left(\binom{n-\alpha_i}{n-k_j}\right)_{i,j=1,\ldots,t}.$$

Proof. The multiplicity $e(R(X,\alpha))$ of $R(X,\alpha)$ is given by the number of disjoint union of paths from P_s to a point of Q_s, $s = 1,\ldots,t$. Let $U_s = (k_s,k_s)$, $k_s \ge \alpha_s$ be a fixed point of Q_s, $s = 1,\ldots,t$. In the following we consider $k_1 < \cdots < k_t$, since otherwise there are no disjoint paths. Thus we get $\mathrm{Paths}(\mathcal{P},\mathcal{U}) = \det(\mathrm{Paths}(P_i,U_j))_{i,j=1,\ldots,t}$ where $\mathcal{P} = P_1,\ldots,P_t$ and $\mathcal{U} = U_1,\ldots,U_t$. But we know that $\mathrm{Paths}(P_i,U_j) = \binom{n-\alpha_i}{n-k_j}$, and the proof is done. □

For the classical rings of symmetric minors $S_t(X)$ it yields the formula

$$e(S_t(X)) = \sum_{1\le k_1<\cdots<k_{t-1}\le n} \det\left(\binom{n-i}{n-k_j}\right)_{i,j=1,\ldots,t-1}.$$

Example 5.2.14 In the particular cases $t = 2, 3, n-2, n-1$ we have

$$\text{(a)}\quad e(S_2(X)) = 2^{n-1}, \qquad \text{(b)}\quad e(S_3(X)) = \binom{2n-3}{n-2},$$

$$\text{(c)}\quad e(S_{n-2}(X)) = \binom{n+2}{6} + \binom{n+3}{6}, \qquad \text{(d)}\quad e(S_{n-1}(X)) = \binom{n+1}{3}.$$

By Theorem 5.2.6 the simplicial complex Δ_α is shellable. Next we show how the shellability passes from a simplicial complex to a subcomplex that fulfill a condition as 5.2.9.

Lemma 5.2.15 *Let Δ be a shellable simplicial complex on a set of vertices V, and V' a subset of V. Suppose that the number $|Z \cap V'|$ does not depend on Z for each facet Z of Δ. The the restriction of Δ to V' is a shellable simplicial complex.*

Proof. Denote by Δ' the restriction of Δ to V', and by d the number $|Z \cap V'|$ where Z is any of Δ. As in Proposition 5.2.10 we get that Δ' is a pure simplicial complex of dimension $d-1$ and that a subset Z' of V' is a facet of Δ' if and only if there exists a facet Z of Δ such that $Z' = Z \cap V'$.

For a facet Z' of Δ' we define

$$\overline{Z'} = \min\{Z : Z \text{ is a facet of } \Delta \text{ with } Z \cap V' = Z'\}$$

where the minimum is taken with respect to the total order on the facets of Δ. Now it is natural to consider a total order on the facets of Δ' by setting

$$Z'_1 < Z'_2 \text{ if and only if } \overline{Z'_1} < \overline{Z'_2}$$

and show that this order gives the desired shelling. Let Z'_1, Z'_2 be facets of Δ' with $Z'_1 < Z'_2$. Then $\overline{Z'_1} < \overline{Z'_2}$, and since the total order on the facets of Δ is a shelling there exists a facet W of Δ, $W < \overline{Z'_2}$ and an element $x \in \overline{Z'_2} \setminus \overline{Z'_1}$ such that $\overline{Z'_2} \setminus W = \{x\}$. Let $W' = W \cap V'$; W' is a facet of Δ' since W is a facet of Δ, and $W' < Z'_2$ by definition. Note that $x \in Z'_2$ since otherwise $W \cap V' = Z'_2$ and $W < \overline{Z'_2}$, a contradiction. Therefore $x \in Z'_2 \setminus Z'_1$ and $Z'_2 \setminus W' = \{x\}$. □

Theorem 5.2.16 *The simplicial complex Δ_δ is shellable.*

Proof. Straightforward by Propositon 5.2.10 and Lemma 5.2.15. □

Theorem 5.2.16 gives an alternative proof of the Cohen-Macaulayness of the ring $R(X, \alpha)$ stated earlier in Proposition 1.0.13(b).

Theorem 5.2.17 *The ring $R(X, \alpha)$ is Cohen-Macaulay.*

Proof. By Theorem 5.2.16 it follows that the complex Δ'_α is shellable, hence $K[\Delta'_\alpha]$ is Cohen-Macaulay by Theorem 5.1.3. Thus we get that the ring $K[X]/\text{in}_\tau(I(X, \alpha))$ is Cohen-Macaulay, and this is sufficient to show that $R(X, \alpha)$ itself is Cohen-Macaulay; see Eisenbud [35]. □

(A) Recall that we have defined a term order τ on the polynomial ring $K[X]$ by setting

$$X_{1n} > X_{1n-1} > \cdots > X_{12} > X_{2n} > X_{2n-1} > \cdots > X_{23} > \cdots > X_{n-1n}.$$

With respect to τ the initial monomial of a pfaffian $\pi = [a_1, \ldots, a_{2t}]$ is $\mathrm{in}_\tau(\pi) = \prod_{i=1}^t X_{a_i a_{2t-i+1}}$. Since we have determined Gröbner bases only for the ideals $I_t(X)$, we restrict ourselves to the classical case. The initial ideal $\mathrm{in}_\tau(I_t(X))$ is the monomial ideal generated by the square-free monomials $X_{a_1 a_{2t}} \cdots X_{a_t a_{t+1}}$ with $1 \le a_1 < \cdots < a_{2t} \le n$. In order to describe its corresponding simplicial complex, let us consider the following subset of the plane

$$V = \{(i,j) : 1 \le i < j \le n\}.$$

On the finite set V we introduce the partial order

$$(i,j) \le (i',j') \text{ if and only if } i \le i' \text{ and } j \le j'.$$

Two pairs (i,j) and (i',j') are incomparable if $i < i'$, $j > j'$ or $i > i'$, $j < j'$. Note that an s-antichain of V is a family of elements $(u_1, v_1), \ldots, (u_s, v_s)$ with the property $u_1 < \cdots < u_s$, $v_1 > \cdots > v_s$, and that it corresponds to the leading term of the $2s$-pfaffian $[u_1, \ldots, u_s, v_s \ldots, v_1]$.

It is now clear that the corresponding simplicial complex of the initial ideal $\mathrm{in}_\tau(I_t(X))$ is

$$\Delta_t = \{Z \subseteq V : Z \text{ does not contain } t\text{-antichains}\},$$

and therefore $R/\mathrm{in}_\tau(I_t(X))$ is the Stanley-Reisner ring of Δ_t. As in part (G), we shall see that the number of the faces of maximal dimension of Δ_t can be interpreted as the number of certain families of non-intersecting paths of V. One observes that a *path* in V is a sequence of points

$$P = (u_1, v_1), \ldots, (u_s, v_s) = Q$$

with $P \ge Q$ and

$$(u_i - u_{i-1}, v_i - v_{i-1}) \text{ equals either } (1,0) \text{ or } (0,1) \text{ for all } i = 2, \ldots, s.$$

Set $P_i = (n - 2i + 1, n)$ and $Q_i = (1, 2i)$, $i = 1, \ldots, t-1$. We shall prove again that the faces of maximal dimension of Δ_t can be described as families of non-intersecting paths from $\mathcal{P} = P_1, \ldots, P_{t-1}$ to $\mathcal{Q} = Q_1, \ldots, Q_{t-1}$. In order to do this we introduce some notation. For a subset Z of V we define

$$b(Z) = \{(i,j) \in Z : \text{ there is no element } (i',j') \in Z \text{ with } i > i' \text{ and } j < j'\}.$$

Note that $b(Z)$ is, in some sense, the right border of Z. It is obvious that $b(Z)$ is always a chain of V. Now set

$$Z_1 = b(Z) \text{ and } Z_i = b\Big(Z \setminus \bigcup_{j=1}^{i-1} Z_j\Big) \text{ for } i > 1.$$

(This is also a light and shadow decomposition of Z, where the light comes from the point $(n-1, n)$). It follows that $Z = \bigcup Z_i$, and Z_i are disjoint chains. We now give another description of the faces of Δ_t in terms of the light and shadow decomposition:

Lemma 5.2.18 *We have* $\Delta_t = \{Z \subseteq V : Z_t = \emptyset\}$.

Proof. This follows readily from the fact that the elements of any antichain of a subset Z of V must belong to different Z_i. □

Theorem 5.2.19 *A subset Z of V is a face of Δ_t of maximal dimension, i.e.* $\dim Z = \dim \Delta_t$ *if and only if Z is a family of non-intersecting paths from $\mathcal{P}$ to $\mathcal{Q}$.*

Proof. Let Z be a face of Δ_t of maximal dimension. By Lemma 5.2.18 we have $Z_t = \emptyset$, hence $Z = \cup_{i=1}^{t-1} Z_i$. Then Z_i must be a path from a point (u_i, n) to a point $(1, v_i)$. We claim that $v_i \geq 2i$ and $u_i \leq n - 2i + 1$. If $v_i < 2i$, then the paths $Z_1, \dots, Z_i$ must pass through the antichain $(1, 2i-1), (2, 2i-2), \dots, (i-1, i+1)$. But this antichain has only $i-1$ elements which have to lie on different paths, a contradiction. Thus we obtain that $|Z_i| = u_i + (n - b_i) \leq 2n - 4i + 1$. Summing up we get

$$|Z| = \sum_{i=1}^{t-1} |Z_i| \leq (t-1)(2n - 2t + 1).$$

Let W_i be the (unique) path from P_i to Q_i passing through the points $(i, 2s+i)$, $(i, 2s + i - 1)$ for $i = n - 2s, \dots, 2$, $s = 1, \dots, t-1$, and consider $W = \cup_{i=1}^{t-1} W_i$. Obviously $W \in \Delta_t$ and $|W| = (t-1)(2n - 2t + 1)$. In particular, we get that

$$\dim \Delta_t = (t-1)(2n - 2t + 1) - 1.$$

Since $\dim Z = \dim \Delta_t$ we must have $|Z_i| = 2n - 4i + 1$ for all i, and it happens only when Z_i is a path from P_i to Q_i.

Conversely, if Z is a family of non-intersecting paths from $\mathcal{P}$ to $\mathcal{Q}$, then these paths are the Z_i in the light and shadow decomposition of Z. Thus $Z_t = \emptyset$, hence $Z \in \Delta_t$ and obviously $\dim Z = \dim \Delta_t$. □

Remark 5.2.20 (a) From the preceding results it follows that any face of Δ_t can be describe as a disjoint union of chains Z_i, $i = 1, \dots, t-1$. Furthermore, we also get that the complex Δ_t is pure.

(b) As a by-product of the proof of Theorem 5.2.19 and Proposition 5.1.1 we can regain the dimension formula of Proposition 1.0.18.

By Theorem 5.2.19 and purity of Δ_t it follows that the multiplicity of $K[\Delta_t]$, and, of course, that of $P_t(X)$ is given by the number of families of non-intersecting paths from $\mathcal{P}$ to $\mathcal{Q}$.

Theorem 5.2.21 *We have*

$$e(P_t(X)) = \det\left(\binom{2n-4t+2}{n-2t-i+j+1} - \binom{2n-4t+2}{n-2t-i-j+1}\right)_{i,j=1,\dots,t}. \quad (5.2)$$

Proof. Set $P_i' = (n-t-i+1, n-t+i+1)$ and $Q_i' = (t-i, t+i)$, $i = 1, \ldots, t-1$. We have that $P_i < P_i' < Q_i' < Q_i$, and note that any path Z_i from P_i to Q_i must pass through P_i' and Q_i'. By using the determinantal formula of Gessel-Viennot [39] we get

$$\text{Paths}(\mathcal{P}, \mathcal{Q}) = \det\left(\text{Paths}(P_i', Q_j')\right)_{i,j=1,\ldots,t-1}.$$

Taking into account that the paths from P_i' to Q_j' are bounded by the line $x = y$ of the plane, we deduce

$$\text{Paths}(P_i', Q_j') = \left(\binom{2n-4t+2}{n-2t-i+j+1} - \binom{2n-4t+2}{n-2t-i-j+1}\right).$$

□

Ghorpade and Krattenthaler [40, Theorem 2] proved that formula (2.5) can be simplified to the following:

Corollary 5.2.22 *We have*

$$e(P_t(X)) = \prod_{1\le i\le j\le n-2t+1} \frac{2t-2+i+j}{i+j}. \tag{5.3}$$

In particular, for $n = 2t+1$ we get $e(P_t(X)) = \frac{t(t+1)(2t+1)}{6}$; see Herzog and Trung [45, pg. 29].

A similar argument to the one of Theorem 5.2.6 shows that the complex Δ_t is shellable. Thus one obtains another proof for the Cohen-Macaulayness of the ring $P_t(X)$.

5.3 Hilbert series of determinantal rings

In order to describe the Hilbert series of determinantal rings we use the combinatorial interpretation of the h-vector of their associated simplicial complex.

(**G**) By virtue of Proposition 5.1.4, Theorem 5.2.6, and Remark 5.2.7 we can interpret the h-vector $h(\Delta_\delta) = (h_0, h_1, \ldots)$ of the complex Δ_δ in terms of families of non-intersecting paths with a given number of right-turns.

If $\mathcal{P} = P_1, \ldots, P_s$ and $\mathcal{Q} = Q_1, \ldots, Q_s$ are two sets of s points in V, let $\text{Paths}(\mathcal{P}, \mathcal{Q})_k$ denote the number of families of non-intersecting paths from $\mathcal{P}$ to $\mathcal{Q}$ with exactly k right-turns. With this notation, we can write $h_k = \text{Paths}(\mathcal{P}, \mathcal{Q})_k$ where $P_i = (a_i, n)$, and $Q_i = (m, b_i)$, $i = 1, \ldots, t$. In particular we have:

Theorem 5.3.1 *The Hilbert series $H_\delta(z)$ of $K[\Delta_\delta]$ and $R(X,\delta)$ is given by*

$$H_\delta(z) = \frac{\sum_k \mathrm{Paths}(\mathcal{P},\mathcal{Q})_k z^k}{(1-z)^d}$$

where $d = (m+n+1)t - \sum_{i=1}^t (a_i+b_i)$ is the Krull dimension, $\mathcal{P} = P_1,\dots,P_s$, $\mathcal{Q} = Q_1,\dots,Q_s$, and $P_i = (a_i,n)$, $Q_i = (m,b_i)$, $i = 1,\dots,t$.

Krattenthaler [52] has found, by purely combinatorial methods, a determinantal formula for the enumeration of families of non-intersecting paths with a given number of right-turns that holds no matter how the starting and end points are located. In fact, he showed that

$$\mathrm{Paths}(\mathcal{P},\mathcal{Q})_k = \sum_{k_1+\cdots+k_t=k} \det\left[\binom{m-a_j-i+j}{k_i}\binom{n-b_i+i-j}{k_i+i-j}\right]_{i,j=1,\dots,t.}$$

If the starting and end points are $\mathcal{P} = (1,n),(2,n),\dots,(t-1,n)$ and $\mathcal{Q} = (m,1),(m,2),\dots,(m,t-1)$ one can show that

$$\sum_k \mathrm{Paths}(\mathcal{P},\mathcal{Q})_k z^k = z^{-\binom{t-1}{2}} \det\left(\sum_k \binom{m-i}{k}\binom{n-j}{k} z^k\right);$$

see Conca and Herzog [26].

(S) Since the facets of the complex Δ'_α are restrictions of the facets of Δ_α to V', it is natural to aim for similar results to the ones of part (G).

In the hypotheses of Lemma 5.2.15 and with the notation introduced in its proof we have:

Lemma 5.3.2 *Let Z' be a facet of Δ'. Then $c(Z') = c(\overline{Z'})$.*

Proof. Let $x \in c(Z')$ and Z'_1 a facet of Δ' such that $Z'_1 < Z'$ and $Z' \setminus Z'_1 = \{x\}$. Then $\overline{Z'_1} < \overline{Z'}$, and by shellability of Δ there exists a facet W of Δ and an element $y \in \overline{Z'}$ such that $W < \overline{Z'}$ and $\overline{Z'} \setminus W = \{y\}$. By definition of $\overline{Z'}$, the restriction of W to V' is not Z'. Thus we get $y = x$ and $x \in c(\overline{Z'})$. For the converse we argue in a similar way. □

Let $P_s = (\alpha_s, n)$ and $Q_s = \{(\alpha_s,\alpha_s),(\alpha_s+1,\alpha_s+1),\dots,(n,n)\}$, $s = 1,\dots,t$. Recall that the set Z' is a facet of Δ'_α if and only if Z' is the disjoint union of $Z'_1,\dots,Z'_t$, where Z'_s is a path from P_s to one of the points of Q_s, $s = 1,\dots,t$. Suppose that Z'_s is a path from (α_s,n) to (k_s,k_s) with $\alpha_s \le k_s$. Define W_s as being the path from (α_s,n) to (n,α_s) obtained from Z'_s by adding the set of points $\{(n,\alpha_s),(n-1,\alpha_s),\dots,(k_s,\alpha_s),(k_s,\alpha_s+1),\dots,(k_s,k_s)\}$. From the definition of shelling on the facets of Δ_α, it is clear that $\overline{Z'}$ is the union of $W_1, \ldots, W_t$. Then by Lemma 5.3.2 we have $c(Z') = c(\overline{Z'})$.

Proposition 5.1.4, Theorem 5.2.16, and Lemma 5.3.2 will give an interpretation of the h-vector $h(\Delta'_\alpha) = (h_0, h_1, \ldots)$ of the complex Δ'_α in terms of families of non-intersecting paths in V' with a given number of right-turns. But first we have to modify the definition of right-turns accordingly. Given a path in V' from a point (a, n) to a point (b, b), and a point (i, j) of the path, then (i, j) is called a *right-turn* of the path if $i < j$ and $(i-1, j)$, $(i, j-1)$ belong to the path, or $i = j$ (in this case $i = b$) and $(i-1, j)$ belong to the path. (Note that the first case correspond to the definition of right-turns in part (G).) It is obvious that the right-turns of Z' are exactly the right-turns of $\overline{Z'}$. Set again $\mathcal{P} = P_1, \ldots, P_t$ and $\mathcal{Q} = Q_1, \ldots, Q_t$ where $P_i = (\alpha_i, n)$, $Q_i = \{(\alpha_i, \alpha_i), (\alpha_i + 1, \alpha_i + 1), \ldots, (n, n)\}$, $i = 1, \ldots, t$. By $\mathrm{Paths}(\mathcal{P}, \mathcal{Q})_k$ we denote the number of families of non-intersecting paths from $\mathcal{P}$ to $\mathcal{Q}$ with exactly k right-turns. Thus we have that $h_k = \mathrm{Paths}(\mathcal{P}, \mathcal{Q})_k$.

Theorem 5.3.3 *The Hilbert series $H_\alpha(z)$ of $K[\Delta'_\alpha]$ and $R(X, \alpha)$ is given by*

$$H_\alpha(z) = \frac{\sum_k \mathrm{Paths}(\mathcal{P}, \mathcal{Q})_k z^k}{(1-z)^d}$$

where $d = (n+1)t - \sum_{i=1}^{t} \alpha_i$ is the Krull dimension.

(**A**) The h-vector of the complex Δ_t can also be interpreted in terms of families of non-intersecting paths. However, the interpretation is not straightforward and we refer the interested reader to Ghorpade and Krattenthaler [40]. Actually they use a determinantal formula of Krattenthaler [52] for the enumeration of families of non-intersecting paths which do not cross the line $x = y$ and with a given number of right-turns in order to prove that

$$\mathrm{Paths}(\mathcal{P}, \mathcal{Q})_k = \sum_k \det\left(\binom{n-2t+2}{k+i-j}\binom{n-2t+2}{k} - \binom{n-2t+1}{k-j}\binom{n-2t+3}{k+i}\right)_{i,j=1,\ldots,t-1}$$

where $\mathcal{P} = \{(t-1, t-1), (t, t-2), \ldots, (2t-3, 1)\}$ and $\mathcal{Q} = \{(n-t+1, n-t+1), (n-t+2, n-t), \ldots, (n-1, n-2t+3)\}$. On the other side, they show that h_k must coincide with $\mathrm{Paths}(\mathcal{P}, \mathcal{Q})_k$. Thus we get:

Theorem 5.3.4 *The Hilbert series $H_t(z)$ of $K[\Delta_t]$ and $P_t(X)$ is given by*

$$H_t(z) = \frac{\sum_k \det\left(\binom{n-2t+2}{k+i-j}\binom{n-2t+2}{k} - \binom{n-2t+1}{k-j}\binom{n-2t+3}{k+i}\right)_{i,j=1,\ldots,t-1}}{(1-z)^{(2n-2t+1)(t-1)}}.$$

5.4 The a-invariant of determinantal rings

For the computation of the a-invariant we restrict our attention to the classical case. Further, we also consider the weighted case.

(**G**) Suppose that $m \le n$ and let $e_1, \dots, e_m$ and $f_1, \dots, f_n$ be two sequences of integers such that $e_i + f_j > 0$ for all i, j. If we give to indeterminate X_{ij} the degree $e_i + f_j$, then all minors of X are homogeneous. Since the ring $R_t(X)$ is invariant under rows and columns permutations we may always assume that $e_1 \le \cdots \le e_m$. Moreover, we order the columns such that $f_1 \le \cdots \le f_n$. In this frame we get the following formula for the a-invariant $a(R_t(X))$ of $R_t(X)$.

Theorem 5.4.1 *We have*

$$a(R_t(X)) = -(t-1)(\sum_{i=1}^{m} e_i + \sum_{j=1}^{n} f_j) - (n-m)\sum_{i=1}^{t-1} e_i \le -(t-1)n.$$

Proof. Denote by Δ_t the complex Δ_δ with $\delta = [1, \dots, t-1|1, \dots, t-1]$. By Proposition 5.1.5 we have that $a(\Delta_t) = -\min\{\rho(Z) : Z \text{ is a facet of } \Delta_t\}$ where

$$\rho(Z) = \sum_{(i,j)\in Z\setminus c(Z)} \deg X_{ij}.$$

Recall that any facet Z of Δ_t is a disjoint union of paths Z_i from $P_i = (i, n)$ to $Q_i = (m, i)$, $i = 1, \dots, t-1$. For Z_i we consider $\rho(Z_i)$ defined identically as above, with the mention that $c(Z_i)$ represent the set of right-turns which belong to Z_i. Consequently $\rho(Z) = \sum_{i=1}^{t-1} \rho(Z_i)$. Note that Z_i necessarily passes through $Q'_i = (m-t+i+1, i)$ and let Z'_i the part of Z_i from P_i to Q'_i. Set $Z' = \cup_{i=1}^{t-1} Z'_i$. Then $\rho(Z)$ and $\rho(Z')$ differs by a constant ω which is independent of Z since none of the elements in $Z_i \setminus Z'_i$ is a right-turn of Z_i. Therefore

$$\min\{\rho(Z) : Z \text{ is a facet of } \Delta_t\} = \min\{\rho(Z')\} + \omega$$

where Z' is the part of Z defined as above.

It is obvious that any path from P_i to Q'_i has at most $m-t+1$ right-turns. Let $\overline{W}_i$ be the path from P_i to Q'_i with exactly $m-t+1$ right-turns, namely $(m-t+i-j+1, i+j+1)$, $j = 0, \dots, m-t$. We claim that $\rho(W_i) \ge \rho(\overline{W}_i)$ for any path W_i from P_i to Q'_i. In order to prove the claim we distinguish three cases.

Case 1. Assume $W_i \subseteq \overline{R}_{\overline{W}_i}$. Then for all j, $i \le j \le n$, there exists a unique k such that $(k, j) \in \overline{W}_i \setminus c(\overline{W}_i)$, and a unique k' with $(k', j) \in W_i$ but

$(k'', j) \notin W_i$ for all $k'' < k'$. Obviously the point (k', j) is not a right-turn of W_i, and thus the map

$$\psi : \overline{W}_i \setminus c(\overline{W}_i) \longrightarrow W_i \setminus c(W_i) \text{ defined by } \psi((k,j)) = (k',j)$$

is a monomorphism. Since $e_k + f_j \leq e_{k'} + f_j$ for all j and k as before we get $\rho(W_i) \geq \rho(\overline{W}_i)$.

Case 2. Assume $W_i \subseteq \overline{L}_{\overline{W}_i}$. Then for all j, $m-t+i+1 \leq j \leq n$, the paths $\overline{W}_i$ and W_i have the point (i,j) in common. Therefore it is enough to compare the parts of the paths $\overline{W}_i$ and W_i starting from $P'_i = (i, m-t+i+1)$ and ending to $Q'_i = (m-t+i+1, i)$. These two points can be considered as the upper left corner, respectively the lower right corner of a square whose edges have length $m-t+1$. In order to simplify notation we may suppose that $P'_i = (1,s)$ and $Q'_i = (s,1)$. We have $\rho(W_i) = \sum_{k=1}^{s} u_k e_k + \sum_{j=1}^{s} v_j f_j$ where u_k and v_j are positive integers. On the other side, we have $\rho(\overline{W}_i) = \sum_{k=1}^{s}(e_k + f_{s-k+1}) = \sum_{k=1}^{s} e_k + \sum_{j=1}^{s} f_j$. Obviously $\rho(W_i) \geq \rho(\overline{W}_i)$.

Case 3. This is the general case when W_i might intersect $\overline{W}_i$ in more points than their starting and ending points. Let W_i^* be a part of W_i in between two successive intersection points, and $\overline{W}_i^*$ the corresponding part of $\overline{W}_i$. Then either $W_i^* \subseteq \overline{R}_{\overline{W}_i^*}$ or $W_i \subseteq \overline{L}_{\overline{W}_i^*}$. According to the first two cases we must have $\rho(W_i^*) \geq \rho(\overline{W}_i^*)$, and summing up we get the desired inequality.

Now the theorem follows easily. Let $\overline{Z}_i = \overline{W}_i \cup \{(m-t+i+2, i), \ldots, (m,i)\}$ and $\overline{Z} = \cup_{i=1}^{t-1} \overline{Z}_i$. It is clear that $\overline{Z}$ is a facet of Δ_t. Then $\rho(Z) \geq \rho(\overline{Z})$ for any facet Z of Δ_t, and thus $a(R_t(X)) = -\rho(\overline{Z})$. Now we can easily derive the desired formula. □

(**S**) If we give to indeterminates X_{ij} the degrees u_{ij}, then all minors of X are homogeneous if and only if $2u_{ij} = u_{ii} + u_{jj}$. Therefore we essentially encounter the following two situations:

(a) There exists $e_1, \ldots, e_n$ positive integers such that $\deg X_{ij} = e_i + e_j$ for all i, j.

(b) There exists $e_1, \ldots, e_n$ non-negative integers such that $\deg X_{ij} = e_i + e_j + 1$ for all i, j.

Since the ring $R_t(X)$ is invariant under rows and columns permutations we may always assume that $e_1 \leq \cdots \leq e_n$. Under these hypotheses we get the following formula for the a-invariant $a(S_t(X))$ of $S_t(X)$.

Theorem 5.4.2 *We have*

$$a(S_t(X)) = \begin{cases} -(t-1)(\sum_{i=1}^n e_i) & \text{if} \quad n \equiv t \bmod(2) \\ \\ -(t-1)(\sum_{i=1}^n e_i) - \sum_{i=1}^{t-1} e_i & \text{if} \quad n \not\equiv t \bmod(2) \end{cases}$$

in the case of degree type (a), respectively

$$a(S_t(X)) = \begin{cases} -(t-1)(\sum_{i=1}^n e_i + \frac{n}{2}) & \text{if} \quad n \equiv t \bmod(2) \\ \\ -(t-1)(\sum_{i=1}^n e_i + \frac{n+1}{2}) & \text{if} \quad n \not\equiv t \bmod(2) \end{cases}$$

in the case of degree type (b).

Proof. The argument is similar to the one of part (G). Denote by Δ_t the complex Δ_α with $\alpha = [1, \ldots, t-1]$. By Proposition 5.1.5 we have that $a(\Delta_t) = -\min\{\rho(Z) : Z \text{ is a facet of } \Delta_t\}$ where

$$\rho(Z) = \sum_{(i,j) \in Z \setminus c(Z)} \deg X_{ij}.$$

Recall that Z is a disjoint union of paths Z_i from $P_i = (i, n)$ to a point of $Q_i = \{(i,i), \ldots, (n,n)\}$, $i = 1, \ldots, t-1$.

The key of the proof is to define a facet W of Δ_t with the property that $\rho(W) \leq \rho(Z)$ for any facet Z of Δ_t. As in part (G), W should be a facet with maximal number of right-turns. Set F_i be the set $\{(i,n), (i, n-1), \ldots, (i, n-t+i+2)\}$ for $i = 1, \ldots, t-2$, and set $F_{t-1} = \emptyset$.

For $n \equiv t \bmod(2)$, we define W_i as being the path from (i,n) to $(\frac{n-t}{2} + i + 1, \frac{n-t}{2} + i + 1)$ obtained from F_i by adding the points $(i, n-t+i+1)$, $(i+1, n-t+i+1)$, $\ldots$, $(i+j, n-t+i-j+1)$, $(i+j+1, n-t+i-j+1)$, $\ldots$, $(i + \frac{n-t}{2}, \frac{n-t}{2} + i + 1)$, $(i + \frac{n-t}{2} + 1, \frac{n-t}{2} + i + 1)$.

If $n \not\equiv t \bmod(2)$, we define W_i to be the path from (i,n) to $(\frac{n-t+1}{2} + i, \frac{n-t+1}{2} + i)$ obtained from F_i by adding the points $(i, n-t+i+1)$, $(i, n-t+i)$, $(i+1, n-t+i)$, $\ldots$, $(i+j, n-t+i-j)$, $(i+j+1, n-t+i-j)$, $\ldots$, $(i + \frac{n-t-1}{2}, \frac{n-t+1}{2} + i)$, $(i + \frac{n-t+1}{2}, \frac{n-t+1}{2} + i)$.

Finally we set $W = \cup_{i=1}^{t-1} W_i$.

In order to prove that $\rho(W) \leq \rho(Z)$ for any facet Z of Δ_t, we start by considering the case $t = 2$ and n even. It is not hard to see that $c(W) = \{(2,n), (3, n-1), \ldots, \frac{n}{2} + 1, \frac{n}{2} + 1)\}$, and so $W \setminus c(W) = \{(1,n), (2, n-1), \ldots, (\frac{n}{2}, \frac{n}{2} + 1)\}$. We have either $\rho(W) = \sum_{i=1}^n e_i$ if the degree is of type (a) or $\rho(W) = \sum_{i=1}^n e_i + \frac{n}{2}$ if the degree is of type (b). Now let Z be a path from $(1,n)$ to (k,k). We observe that if $i < k$ (resp. $i \geq k$) then there exists j such that $(i,j) \in Z$ (resp. $(j,i) \in Z$), and if $(i,j) \in c(Z)$ then $(i, j-1) \in Z \setminus c(Z)$ (resp. if $(j,i) \in c(Z)$ then $(j-1, i) \in Z \setminus c(Z)$). Thus we get readily that $\rho(W) \leq \rho(Z)$.

If $t = 2$ and n odd, we have either $\rho(W) = \sum_{i=1}^{n} e_i + e_1$ if the degree is of type (a) or $\rho(W) = \sum_{i=1}^{n} e_i + e_1 + \frac{n+1}{2}$ if the degree is of type (b). Let Z be a path from $(1, n)$ to (k, k). By the same argument as above one deduces that $|Z \backslash c(Z)| \geq \frac{n+1}{2}$, and that there exists an i which appears twice as a coordinate of some elements in $Z \setminus c(Z)$. Taking into account that $e_1 \leq \cdots \leq e_n$ we get $\rho(W) \leq \rho(Z)$.

In the general case, let us consider $Z = \cup_{k=1}^{t-1} Z_k$ be a facet of Δ_t. Since the paths Z_k are non-intersecting, we must have $F_k \subseteq Z_k$ for all k. We may consider Z_k and W_k as paths starting from $(i, n-t+i+1)$, and argue as before to show that $\rho(W_k) \leq \rho(Z_k)$ for all k. Thus we get

$$\rho(W) = \sum_{k=1}^{t-1} \rho(W_k) \leq \sum_{k=1}^{t-1} \rho(Z_k) = \rho(Z)$$

and the proof is done. □

(**A**) Let $e_1, \ldots, e_n$ be a sequence of non-negative integers such that $e_i \equiv e_j \mod(2)$ for all i, j. Set $\deg X_{ij} = \frac{1}{2}(e_i + e_j)$. Thus all pfaffians of X are homogeneous. We get the following formula for the a-invariant $a(R_t(X))$ of $R_t(X)$.

Theorem 5.4.3 *We have*

$$a(P_t(X)) = \begin{cases} -(t-1)(\sum_{i=1}^{n} e_i) & \textit{if} \quad 2t-2 < n, \\ \\ -(t-\frac{3}{2})(\sum_{i=1}^{n} e_i) & \textit{if} \quad 2t-2 = n. \end{cases}$$

Proof. The case $2t - 2 = n$ is trivial. Assume that $2t - 2 < n$ and consider the facet Z of Δ_t with

$$Z_i = \{(j, 2i+j-1) : j = 1, \ldots, n-2i+1\} \cup \{(j, 2i+j) : j = 1, \ldots, n-2i\}$$

where $i = 1, \ldots, t-1$. One observes that Z is maximal in the shelling order. For this facet we have $c(Z) = \{(l-k, l+k+1) : k = 0, \ldots, t-2 \text{ and } l = t, \ldots, n-t\}$. We get that $b = \sum_{(i,j) \in Z \backslash c(Z)} \frac{1}{2}(e_i + e_j) = (t-1) \sum_{i=1}^{n} e_i$. On the other side, one checks easily that the corresponding number b for any facet of Δ_t is greater or equal than $(t-1) \sum_{i=1}^{n} e_i$. The desired formula follows readily by using Proposition 5.1.5. □

Chapter 6

Algebras of minors

Let I be an ideal of a Noetherian ring R. The Rees algebra of I is the graded R-algebra $\mathcal{R}(I) = \bigoplus_{k=0}^{\infty} I^k$. When I is a prime ideal there is another graded R-algebra associated to I, namely the symbolic Rees algebra $\mathcal{R}^s(I) = \bigoplus_{k=0}^{\infty} I^{(k)}$. In this chapter we treat a special and interesting case: the Rees algebras of determinantal (pfaffian) ideals and their special fibers. To enter the details, we are going to investigate three types of graded algebras associated to determinantal (pfaffian) ideals: the Rees algebra $\mathcal{R}(I)$ of a product $I = I_{t_1}(X) \cdots I_{t_u}(X)$ of determinantal (pfaffian) ideals (with emphasis on the case $u = 1$), the symbolic Rees algebra $\mathcal{R}^s(I_t)$ of $I_t(X)$, and the algebra of t-minors ($2t$-pfaffians) A_t, that is the subalgebra of $K[X]$ generated by the t-minors ($2t$-pfaffians) of X. First we are going to do a detour by studying their initial algebras. In all three cases the initial algebra is a normal affine semigroup ring and its description is simply a translation of the results of Chapter 4 into the algebra setting.

6.1 Cohen-Macaulayness and normality of algebras $\mathcal{R}(I_t)$ and A_t

(G) Let $X = (X_{ij})$ be a generic matrix of size $m \times n$ over a field K, and assume for simplicity that $m \leq n$. In the following we consider A_t and $\mathcal{R}(I_t)$ as subalgebras of a common polynomial ring $S = K[X,T] = K[X][T]$ obtained by adjoining a new indeterminate T to $K[X]$, A_t being generated by the elements of M_tT, where M_t is the set of t-minors, and $\mathcal{R}(I_t)$ being generated by M_tT and the entries of X.

For $t = 1$, the Rees algebra of the polynomial ring $K[X]$ with respect to the ideal $I_1(X)$ can be represented as a determinantal ring, which is known as being a normal Cohen-Macaulay domain. For A_t the case $t = 1$ is quite trivial, since $A_1 = K[X]$. In the opposite extreme case of maximal minors, i.e. $t = m$, Eisenbud and Huneke [36] proved that $\mathcal{R}(I_t)$ is normal and Cohen-Macaulay. Their approach is based on the notion of algebra with straightening law. On the other side, the algebra A_m is nothing but the homogeneous coordinate ring of the Grassmann variety which is known to be a Gorenstein factorial domain; for instance, see Bruns and Vetter [21]. For A_t we get another simple case, revealed by its dimension: if $t < m$, then $\dim A_t = \dim K[X] = mn$ (see Bruns and Vetter [21, Prop. (10.16)]) and so, for $t = m - 1 = n - 1$ the algebra A_t is generated by $mn = \dim A_t$ elements, therefore it is isomorphic to a polynomial ring over K. In the general case Bruns proved that $\mathcal{R}(I_t)$ and A_t are Cohen-Macaulay and normal for any t provided $\operatorname{char} K = 0$, by using invariant theory methods; see Bruns [15]. In this part we extend Bruns' results to the positive characteristic case; see Bruns and Conca [16].

We say that K has *non-exceptional characteristic* if either $\operatorname{char} K = 0$ or $\operatorname{char} K > \min(t, m-t, n-t)$. For exceptional characteristic, $\mathcal{R}(I_t)$ and A_t can be (and perhaps are always) very far from being Cohen-Macaulay.

Example 6.1.1 Let K be a field of characteristic 2 and suppose that $m, n \geq 4$. Set $I = I_2(X)$ and note that $\dim \mathcal{R}(I) = mn + 1$. Now we can use the computer program MACAULAY [9] to show that $[1|1][234|234] \notin I^2$. On the other hand, it is not difficult to see that $[34|34]([1|1][234|234]) \in I^3$. Starting with these observations Bruns [15, Prop. (4.1)] proved that $\operatorname{depth} \mathcal{R}(I) = 1$. In particular, Rees algebra $\mathcal{R}(I)$ is not a Cohen-Macaulay ring. Moreover, in this case A_2 is not normal. The product $[1|1][234|234]$ is integral over I^2, hence over A_2 and belongs to the field of fractions of S_2.

The best result we can hope for is that $\mathcal{R}(I_t)$ and A_t are Cohen-Macaulay in non-exceptional characteristic. As we have seen, the extreme cases $t = 1$ and $t = m$ have nothing to do with the characteristic of K. Therefore we will concentrate our attention to the remaining cases, that is, we make the following

Assumptions. (a) When studying the Rees algebra $\mathcal{R}(I_t)$, we will assume that $t < \min(m, n)$,

(b) when studying A_t, we will assume that $1 < t < \min(m, n)$ and that $m \neq n$ if $t = \min(m, n) - 1$.

Let us recall that powers and products of determinantal ideals are intersections of symbolic powers; see Theorem 4.1.3 and Corollary 4.1.4. It follows that the Rees algebra of a product $I = I_{t_1}(X) \cdots I_{t_u}(X)$ is the intersection of symbolic Rees algebras of the various $I_t(X)$ and their Veronese subalgebras. Then Lemma 4.3.5 implies that the description as an intersection passes on to the initial algebras. All these observations show that we have to focus our attention on the description of the initial algebra of the symbolic Rees algebra $\mathcal{R}^s(I_t)$.

First note that the description of the symbolic powers of determinantal ideals in Theorem 4.1.1 yields a description of the symbolic Rees algebra given by

$$\mathcal{R}^s(I_t) = K[X][I_t(X)T, I_{t+1}(X)T^2, \ldots, I_m(X)T^{m-t+1}].$$

Let τ be a diagonal term order on $K[X]$ and extend it arbitrarily to a term order on S. Then the initial algebra $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ of $\mathcal{R}^s(I_t)$ is given by

$$\bigoplus_{k=0}^{\infty} \operatorname{in}_\tau(I_t(X)^{(k)})T^k.$$

Application of Theorem 4.3.4 provides the following

Proposition 6.1.2 *One has*

$$\operatorname{in}_\tau(\mathcal{R}^s(I_t)) = K[X][\operatorname{in}_\tau(I_t(X))T, \operatorname{in}_\tau(I_{t+1}(X))T^2, \ldots, \operatorname{in}_\tau(I_m(X))T^{m-t+1}].$$

In particular, a monomial MT^k is in $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ if and only if $\hat{\gamma}_t(M) \geq k$.

In order to prove that the symbolic Rees algebra is Cohen-Macaulay and normal we need to show that KRS commutes with taking the powers of (bi)tableaux.

Lemma 6.1.3 *Let Σ be a standard (bi)tableau. Then*

$$\operatorname{KRS}(\Sigma^k) = \operatorname{KRS}(\Sigma)^k,$$

for all positive integers k. In particular, for any monomial M of $K[X]$ and any positive integer k we have $\hat{\gamma}_t(M^k) = k\hat{\gamma}_t(M)$.

Proof. First observe that a power of a standard (bi)tableau is again standard. Then note that k successive applications of DELETE algorithm on Σ^k act like a single application on k copies of Σ. For the second part, we just recall that we can write $M = \operatorname{KRS}(\Sigma)$ with Σ a standard bitableau, and by Theorem 4.3.2 $\hat{\gamma}_t(M) = \gamma(\Sigma)$. □

As we already mentioned at the beginning of this chapter, we prove that the initial algebra $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ is a normal affine semigroup ring.

Theorem 6.1.4 *The initial algebra $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ of the symbolic Rees algebra $\mathcal{R}^s(I_t)$ is finitely generated and normal. In particular, $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ and $\mathcal{R}^s(I_t)$ are Cohen-Macaulay.*

Proof. By Theorem 4.3.4 it follows that the monomial algebra $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ is finitely generated. For normality it suffices to show that $MT^l \in \operatorname{in}_\tau(\mathcal{R}^s(I_t))$ whenever a power of MT^l belongs to $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$; see Bruns and Herzog [19, Theorem 6.1.4]. Assume that $(MT^l)^k \in \operatorname{in}_\tau(\mathcal{R}^s(I_t))$. By virtue of Theorem 6.1.2 we have $\hat{\gamma}_t(M^k) \geq kl$. On the other side, from Lemma 6.1.3 we have $\hat{\gamma}_t(M^k) = k\hat{\gamma}_t(M)$, and therefore $\hat{\gamma}_t(M) \geq l$. Using again Theorem 6.1.2 we get $MT^l \in \operatorname{in}_\tau(\mathcal{R}^s(I_t))$.

From Hochster's theorem, the fact that $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ is Cohen-Macaulay follows from its normality; see Bruns and Herzog [19, Theorem 6.3.5]. For the Cohen-Macaulay-ness of the symbolic Rees algebra we use a result of Conca, Herzog, and Valla [27, Theorem 2.3]. □

More general, for products of determinantal ideals we get:

Theorem 6.1.5 *Let $\sigma = (t_1, \dots, t_u)$ be a non-increasing sequence of integers. Set $g_i = \gamma_i(\sigma)$, $I = I_{t_1}(X) \cdots I_{t_u}(X)$ and suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t_i, m - t_i, n - t_i)$ *for all $i = 1, \dots, u$. Then*
(a) $\operatorname{in}_\tau(\mathcal{R}(I))$ is finitely generated and normal,
(b) $\mathcal{R}(I)$ is Cohen-Macaulay and normal.

Proof. As already mentioned $\operatorname{in}_\tau(\mathcal{R}(I)) = \bigoplus_{k=0}^{\infty} \operatorname{in}_\tau(I^k)T^k$, and by Theorem 4.3.6 $\operatorname{in}_\tau(I) = \bigcap_{i=1}^{t_1} \operatorname{in}_\tau(I_i(X)^{(g_i)})$. Therefore we have

$$\operatorname{in}_\tau(\mathcal{R}(I)) = \bigcap_{i=1}^{t_1} \bigoplus_{k=0}^{\infty} \operatorname{in}_\tau(I_i(X)^{(kg_i)})T^k.$$

Observe that for $g_i > 0$ the monomial algebra $\bigoplus_{k=0}^{\infty} \operatorname{in}_\tau(I_i(X)^{(kg_i)})T^k$ is isomorphic to the g_ith Veronese subalgebra of the monomial algebra $\operatorname{in}_\tau(\mathcal{R}^s(I_i))$, while for $g_i = 0$ it equals the polynomial ring S. By Theorem 6.1.4 we know that the algebra $\operatorname{in}_\tau(\mathcal{R}^s(I_i))$ is finitely generated and normal, hence $\bigoplus_{k=0}^{\infty} \operatorname{in}_\tau(I_i(X)^{(kg_i)})T^k$ is finitely generated and normal. Since the intersection of a finite number of finitely generated normal monomial algebras is finitely generated and normal (see Bruns and Herzog [19, Prop. 6.1.2 and Theorem 6.1.4]), we thus get that $\operatorname{in}_\tau(\mathcal{R}(I))$ is a normal finitely generated monomial algebra.

For (b) we use the same argument as in the proof of Theorem 6.1.4. □

In particular, we obtain the following:

Corollary 6.1.6 *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t, m - t)$. *Then the Rees algebra $\mathcal{R}(I_t)$ is Cohen-Macaulay and normal.*

For the algebra of minors A_t we have:

Theorem 6.1.7 *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t, m-t)$. *Then the initial algebra* $\operatorname{in}_\tau(A_t)$ *is finitely generated and normal. In particular,* A_t *is Cohen-Macaulay and normal.*

Proof. In Section 6.2 we shall denote by V_t the K-subalgebra of S generated by the monomials MT with $\deg M = t$. Note that $A_t \subset V_t$ and that V_t is a normal semigroup ring isomorphic to the t-Veronese subalgebra of the polynomial ring S. Furthermore, $A_t = \mathcal{R}(I_t) \cap V_t$ and this implies that $\operatorname{in}_\tau(A_t) = \operatorname{in}_\tau(\mathcal{R}(I_t)) \cap V_t$. By Theorem 6.1.5 the initial algebra $\operatorname{in}_\tau(\mathcal{R}(I_t))$ is normal, and V_t is obviously normal, so we get that $\operatorname{in}_\tau(A_t)$ is normal. □

(**A**) Let $X = (X_{ij})$ be a generic alternating matrix of size n over a field K. As in part (G), we can consider A_t and $\mathcal{R}(I_t)$ as subalgebras of the same polynomial ring $S = K[X,T] = K[X][T]$ obtained by adjoining a new indeterminate T to $K[X]$, A_t being generated by the elements of M_tT, where M_t is the set of $2t$-pfaffians, and $\mathcal{R}(I_t)$ being generated by M_tT and the entries of X.

For $t = 1$, Rees algebra can be again represented as a determinantal ring, and thus it is a normal Cohen-Macaulay domain, while $A_1 = K[X]$. In the case of maximal pfaffians we can have: (1) n even and $2t = n$; in this case everything is trivial, or (2) n odd and $2t = n-1$; in this case Eisenbud and Huneke [36, Prop. 2.8] proved that $\mathcal{R}(I_t)$ is normal and Cohen-Macaulay, while A_t is isomorphic to a polynomial ring over K in n indeterminates. Once more, for A_t we get another simple case, revealed again by its dimension: if $2t < n-1$, then $\dim A_t = \dim K[X] = \binom{n}{2}$ (see De Negri [32, Prop. 1.1]) and so, for n even and $2t = n-2$ the algebra A_t is generated by $\binom{n}{2} = \dim A_t$ elements, therefore it is isomorphic to a polynomial ring over K. In general, Bruns [15] hinted that $\mathcal{R}(I_t)$ and A_t are Cohen-Macaulay and normal for any t as long as $\operatorname{char} K = 0$, by employing invariant theory methods. In this part we extend Bruns' results to the positive characteristic case; see Baetica [5].

We say that K has *non-exceptional characteristic* if either $\operatorname{char} K = 0$ or $\operatorname{char} K > \min(2t, n-2t)$. For exceptional characteristic, $\mathcal{R}(I_t)$ and A_t can be (and perhaps are always) very far from being Cohen-Macaulay.

Example 6.1.8 Let us consider $n = 8$ and $\operatorname{char} K = 3$. Set $I = I_2(X)$ and note that $\dim \mathcal{R}(I) = 29$. A run of the computer program MACAULAY [9] reveals that [12][345678] does not belong to I^2. On the other side, it is easy to show that $[5678]([12][345678]) \in I^3$. Similarly to Bruns [15, Prop. (4.1)] we can show that $\operatorname{depth} \mathcal{R}(I) = 1$, hence $\mathcal{R}(I)$ is not Cohen-Macaulay. Moreover, in this case A_2 is not normal. The product [12][345678] is integral over I^2, hence over A_2 and belongs to the field of fractions of S_2.

Therefore we have to restrict our attention to the non-exceptional characteristic case. As we have seen before, the extreme cases are rather easy.

Therefore we will concentrate our attention to the remaining cases, that is, when studying the Rees algebra $\mathcal{R}(I_t)$ we assume that $2t < n$, and when studying A_t we assume that $2 < 2t < n-2$.

In Chapter 4 we have proved that the powers and products of pfaffian ideals are intersections of symbolic powers; see Theorem 4.1.15 and Corollary 4.1.16. As in part (G) we will start with the description of the initial algebra of the symbolic Rees algebra $\mathcal{R}^s(I_t)$.

In this case Theorem 4.1.1 yields a description of the symbolic Rees algebra given by

$$\mathcal{R}^s(I_t) = K[X][I_t(X)T, I_{t+1}(X)T^2, \dots, I_m(X)T^{m-t+1}],$$

where $m = [n/2]$. Let τ be the diagonal term order on $K[X]$ defined in Chapter 3 and extend it arbitrarily to a term order on S. A direct application of Theorem 4.3.11 provides the following

Proposition 6.1.9 *One has*

$$\operatorname{in}_\tau(\mathcal{R}^s(I_t)) = K[X][\operatorname{in}_\tau(I_t(X))T, \operatorname{in}_\tau(I_{t+1}(X))T^2, \dots, \operatorname{in}_\tau(I_m(X))T^{m-t+1}],$$

with $m = [n/2]$.

In particular, a monomial MT^k is in $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ if and only if $\hat{\gamma}_t(M) \geq k$.

Since KRS commutes with taking the powers of tableaux we get:

Lemma 6.1.10 *For any monomial M of $K[X]$ and any positive integer k we have $\hat{\gamma}_t(M^k) = k\hat{\gamma}_t(M)$.*

Proof. Recall that we can write $M = \operatorname{KRS}(P)$ with P a standard tableau, and by Theorem 4.3.9 we get $\hat{\gamma}_t(M) = \gamma(P)$. □

Similarly to Theorem 6.1.4 we can establish the following:

Theorem 6.1.11 *The initial algebra $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ of the symbolic Rees algebra $\mathcal{R}^s(I_t)$ is finitely generated and normal. In particular, $\operatorname{in}_\tau(\mathcal{R}^s(I_t))$ and $\mathcal{R}^s(I_t)$ are Cohen-Macaulay.*

More general, for products of pfaffian ideals we get:

Theorem 6.1.12 *Let $\sigma = (t_1, \dots, t_u)$ be a non-increasing sequence of integers. Set $g_i = \gamma_i(\sigma)$, $I = I_{t_1}(X) \cdots I_{t_u}(X)$ and suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(2t_i, n - 2t_i)$ *for all $i = 1, \dots, u$. Then*

(a) $\operatorname{in}_\tau(\mathcal{R}(I))$ *is finitely generated and normal,*

(b) $\mathcal{R}(I)$ *is Cohen-Macaulay and normal.*

In particular, we obtain the following:

Corollary 6.1.13 *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(2t, n-2t)$. *Then the Rees algebra* $\mathcal{R}(I_t)$ *is Cohen-Macaulay and normal.*

For the algebra of pfaffians A_t we have

Theorem 6.1.14 *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(2t, n-2t)$. *Then the initial algebra* $\operatorname{in}_\tau(A_t)$ *is finitely generated and normal. In particular,* A_t *is Cohen-Macaulay and normal.*

Proof. Similar to 6.1.7. □

6.2 Divisor class group of $\mathcal{R}(I_t)$ and A_t

(**G**) For maximal minors the divisor class group and the canonical class of $\mathcal{R}(I_t)$ have been determined. They are the "expected" ones, that is, those of the Rees algebra of a prime ideal with primary powers in a regular local ring: $\operatorname{Cl}(\mathcal{R}(I_t))$ is free of rank 1 with generator $\operatorname{cl}(I_t\mathcal{R}(I_t))$ and the canonical class is $(2 - \operatorname{height} I_t)\operatorname{cl}(I_t\mathcal{R}(I_t))$; see Herzog and Vasconcelos [44], or Bruns, Simis and Trung [20]. On the other hand, for non-maximal minors, $\operatorname{Cl}(\mathcal{R}(I_t))$ is free of rank t; see Bruns [15].

Our goal is to describe the divisor class group of $\mathcal{R}(I_t)$ and A_t for $t < \min(m, n)$. The main tools are the γ-functions that allow us to describe all relevant ideals in $\mathcal{R}(I_t)$ and A_t and the Sagbi deformation by which we will lift the canonical module from the initial algebras of $\mathcal{R}(I_t)$ and A_t to the algebras $\mathcal{R}(I_t)$ and A_t themselves.

From now on we will always assume that the characteristic of the field K is non-exceptional. Clearly, $A_t \subset \mathcal{R}(I_t)$ and both algebras are $\mathbb{N}$-graded in a natural way. Moreover $\mathcal{R}(I_t)/(X)\mathcal{R}(I_t) \cong A_t$, where $(X)\mathcal{R}(I_t)$ is the ideal of $\mathcal{R}(I_t)$ generated by the X_{ij}. Denote by V_t the K-subalgebra of S generated by monomials that have degree t in the variables X_{ij} and degree 1 in T. Note that $A_t \subset V_t$ and that V_t is a normal semigroup ring isomorphic to the t-Veronese subalgebra of the polynomial ring S. The rings $\mathcal{R}(I_t)$ and A_t are already known to be normal Cohen-Macaulay domains. The aim of this part is to determine their divisor class group and canonical class and to discuss the Gorenstein property of these rings. As we have mentioned before these invariants are already known is some special cases. Therefore we will concentrate our attention to the remaining cases.

Our approach to the study of the algebras under investigation makes use of Sagbi bases deformations and of the straightening law for generic minors. For generalities on the former we refer the reader to Conca, Herzog and Valla [27].

We may define the value of function γ_t for any polynomial f of $K[X]$ as follows. Let $f = \sum_{i=1}^{p} \lambda_i \mu_i$ the unique representation of f as a linear combination of standard monomials μ_i with coefficients $\lambda_i \neq 0$. Then we set

$$\gamma_t(f) = \inf\{\gamma_t(\mu_i) : i = 1, \ldots, p\}$$

and $\gamma_t(0) = +\infty$. This definition is consistent with the one above in the sense that if μ is a product of minors of shape s which is non-standard, then $\gamma_t(\mu) = \gamma_t(s)$.

The function γ_t is indeed a discrete valuation on $K[X]$ (with values in $\mathbb{N}$), that is, the following conditions are satisfied for every f and g in $K[X]$:

(a) $\gamma_t(f + g) \geq \min(\gamma_t(f), \gamma_t(g))$, and equality holds if $\gamma_t(f) \neq \gamma_t(g)$,

(b) $\gamma_t(fg) = \gamma_t(f) + \gamma_t(g)$.

Note that (a) follows immediately from the definition, and (b) from the fact that the associated graded ring of a symbolic filtration is a domain if the base ring is regular. We may further extend γ_t to the field of fractions $Q(K[X])$ of $K[X]$ by setting

$$\gamma_t(f/g) = \gamma_t(f) - \gamma_t(g)$$

so that S is a subring of the valuation ring associated with each γ_t.

The next step is to extend the valuation γ_t to the polynomial ring S and to its field of fractions. We want to do this in a way such that the subalgebras $\mathcal{R}(I_t)$ and A_t of S will then be described in terms of these functions. So, from now on, let us fix a number t with $1 \leq t \leq \min(m, n)$. For every polynomial $F = \sum_{j=0}^{p} f_j T^j \neq 0$ of S we set

$$\gamma_i(F) = \inf\{\gamma_i(f_j) - j(t + 1 - i) : j = 0, \ldots, p\};$$

in particular

$$\gamma_i(T) = -(t + 1 - i).$$

Then we have:

Proposition 6.2.1 *The function γ_i defines a discrete valuation on the field of fractions $Q(S)$ of S.*

Proof. This is a general fact, see Bourbaki [10, Ch. VI, §10, no. 1, lemme 1]. □

Note that $F = \sum f_j T^j \in S$ has $\gamma_i(F) \geq 0$ if and only if $f_j \in I_i(X)^{(j(t+1-i))}$ for every j. It follows from Corollary 4.1.4 that the Rees algebra $\mathcal{R}(I_t)$ of $I_t(X)$ has the following description:

Lemma 6.2.2 *One has*

$$\mathcal{R}(I_t) = \{F \in S : \gamma_i(F) \geq 0 \text{ for } i = 1, \ldots, t\}.$$

Similarly $A_t = \{F \in V_t : \gamma_i(F) \geq 0 \text{ for } i = 2, \ldots, t\}$. But this description is redundant:

Lemma 6.2.3 *One has*

$$A_t = \{F \in V_t : \gamma_2(F) \geq 0\}.$$

Proof. We have to show that if μ is a standard monomial with $\deg \mu = kt$ and $\gamma_2(\mu) \geq k(t-1)$, then $\gamma_i(\mu) \geq k(t+1-i)$ for all $j = 3, \ldots, t$. For $i = 1, \ldots, m$ let a_i denote the number of the factors of μ which are minors of size i. By assumption we have that

$$\text{(a)} \quad \sum_{i=1}^{m} i a_i = kt \qquad \text{and} \qquad \text{(b)} \quad \sum_{i=2}^{m} (i-1) a_i \geq k(t-1).$$

In view of (a), condition (b) can be rewritten as

$$\text{(c)} \quad \sum_{i=1}^{m} a_i \leq k.$$

We have to show that

$$\text{(d)} \quad \sum_{i=j}^{m} a_i (i + 1 - j) \geq k(t + 1 - j).$$

Note that

$$\textstyle\sum_{i=j}^{m} a_i(i+1-j) = \sum_{i=1}^{m} a_i(i+1-j) - \sum_{i=1}^{j-1} a_i(i+1-j) =$$
$$kt + \textstyle\sum_{i=1}^{m} a_i(1-j) + \sum_{i=1}^{j-1} a_i(j-i-1).$$

Therefore (d) is equivalent to

$$\Big(k - \sum_{i=1}^{m} a_i\Big)(j-1) + \sum_{i=1}^{j-1} a_i(j-i-1) \geq 0$$

which is true since the left hand side is the sum of non-negative terms. □

Similarly the initial algebras of $\mathcal{R}(I_t)$ and A_t have a description in terms of the functions $\hat{\gamma}_i$. To this end we extend the definition of the function $\hat{\gamma}_i$ to monomials of S by setting

$$\hat{\gamma}_i(MT^k) = \hat{\gamma}_i(M) - k(t + 1 - i)$$

where M is a monomial of S. Furthermore, for a polynomial $F = \sum_{j=1}^{p} \lambda_j N_j$ of S where the N_i are monomials and the λ_j are non-zero elements of K, we set

$$\hat{\gamma}_i(F) = \inf\{\hat{\gamma}_i(N_j) : j = 1, \dots, p\}.$$

Now Theorem 4.1.4 implies:

Lemma 6.2.4 (a) *The initial algebra* $\operatorname{in}_\tau(\mathcal{R}(I_t))$ *of* $\mathcal{R}(I_t)$ *is given by*

$$\operatorname{in}_\tau(\mathcal{R}(I_t)) = \{F \in S : \hat{\gamma}_i(F) \geq 0 \text{ for } i = 1, \dots, t\}.$$

The set of the monomials N of S such that $\hat{\gamma}_i(N) \geq 0$ for $i = 1, \dots, t$ form a K-vector space basis of $\operatorname{in}_\tau(\mathcal{R}(I_t))$.

(b) *The initial algebra* $\operatorname{in}_\tau(A_t)$ *of A_t is given by*

$$\operatorname{in}_\tau(A_t) = \{F \in V_t : \hat{\gamma}_2(F) \geq 0\}.$$

The set of the monomials N of S such that $\hat{\gamma}_1(N) = 0$ and $\hat{\gamma}_2(N) \geq 0$ form a K-vector space basis of $\operatorname{in}_\tau(A_t)$.

The major difference between the functions γ_i and $\hat{\gamma}_i$ is that the latter is not a valuation. Nevertheless, it will turn out that $\hat{\gamma}_i$ is an "intersection" of valuations; see Section 6.3.

The divisor class group of $\mathcal{R}(I_t)$ can be determined by the primary decomposition of the divisorial ideal $I_t(X)\mathcal{R}(I_t)$ using a theorem of Simis and Trung [59, Theorem 1.1]. We present a (slightly) different approach to it, which will also be used to determine the divisor class group of A_t.

For $i = 1, \dots, t$ we set $P_i = \{F \in \mathcal{R}(I_t) : \gamma_i(F) \geq 1\}$. Since γ_i is a discrete valuation, it follows that P_i is a prime ideal of $\mathcal{R}(I_t)$. Furthermore:

Lemma 6.2.5 *For every $i = 1, \dots, t$ and $j > 0$ one has*

$$P_i^{(j)} = \{F \in \mathcal{R}(I_t) : \gamma_i(F) \geq j\}.$$

Proof. Set $P_i(j) = \{F \in \mathcal{R}(I_t) : \gamma_i(F) \geq j\}$. By construction, $P_i(j)$ is P_i-primary and $P_i(j) \supset P_i^j$. So it is enough to show that $P_i(j) \subseteq P_i^{(j)}$, or, in other words, that for every $F \in P_i(j)$ there exists $G \in \mathcal{R}(I_t)$ such that $\gamma_i(G) = 0$ and $GF \in P_i^j$. We may assume that $F = \mu T^k$ where μ is a product of minors. Let δ be a minor of size $i-1$ (with $\delta = 1$ if $i = 1$). Evaluating the γ-functions and using Theorem 4.1.3, one then concludes $\delta^{2j}\mu \in I_i(X)^j I_t(X)^k$. This implies that $\delta^{2j}\mu T^k \in (I_i(X)\mathcal{R}(I_t))^j$. As $I_i(X)\mathcal{R}(I_t) \subset P_i$ and $\gamma_i(\delta) = 0$, we are done. □

The ideal $I_t(X)\mathcal{R}(I_t)$ is a height 1 ideal of $\mathcal{R}(I_t)$. This follows, for instance, from the fact that $\mathcal{R}(I_t)$ has dimension $mn + 1$ and associated graded ring $\mathcal{R}(I_t)/I_t(X)\mathcal{R}(I_t)$ has dimension mn. Furthermore $I_t(X)\mathcal{R}(I_t)$ is a divisorial ideal since it is isomorphic to $\bigoplus_{k=1}^{\infty} I_t(X)^k T^k$, which is a height 1 prime ideal.

As a consequence of Lemma 6.2.5 and Corollary 4.1.4 we obtain

Proposition 6.2.6 *One has*

$$I_t(X)\mathcal{R}(I_t) = \cap_{i=1}^t P_i^{(t-i+1)},$$

and this is an irredundant primary decomposition of $I_t(X)\mathcal{R}(I_t)$. In particular, P_i is a height 1 prime ideal of $\mathcal{R}(I_t)$ for $i = 1, \dots, t$.

Proof. By virtue of Lemma 6.2.5 the equality $I_t(X)\mathcal{R}(I_t) = \cap_{i=1}^t P_i^{(t-i+1)}$ is simply a re-interpretation of Corollary 4.1.4. Since we assume that $t < \min(m, n)$, the primary decomposition of $I_t(X)^k$ given in Corollary 4.1.4 is irredundant for $k \gg 0$. It follows that the primary decomposition of $I_t(X)\mathcal{R}(I_t)$ is also irredundant. □

Remark 6.2.7 Since one can prove directly that P_1 is a height 1 prime ideal with primary powers, one can also use the standard localization argument in order to show Lemma 6.2.5 and Proposition 6.2.6 (see Bruns [15]).

Proposition 6.2.8 *One has*

$$S = \cap \mathcal{R}(I_t)_Q$$

where the intersection is extended over all the height 1 prime ideals Q of $\mathcal{R}(I_t)$ different from $P_1, \dots, P_t$.

Proof. We have to show that for every height 1 prime ideal P of S the intersection $P \cap \mathcal{R}(I_t)$ is a height 1 prime ideal of $\mathcal{R}(I_t)$. Let δ be a t-minor. Then it is easy to see that $\mathcal{R}(I_t)[\delta^{-1}] = S[\delta^{-1}]$. Since taking intersections commutes with localizations, it is enough to deal with the height 1 prime ideals of S which contain δ. But δ is a prime element in S, so that we may assume $P = (\delta)$. Set $Q = \mathcal{R}(I_t) \cap (\delta)$, that is, $Q = \{a\delta \in \mathcal{R}(I_t) : a \in S\}$. We claim

$$(\delta)\mathcal{R}(I_t) = Q \cap P_1^{(t)} \cap P_2^{(t-1)} \cap \dots \cap P_t. \tag{1}$$

The inclusion $\subseteq$ is trivial. For the other inclusion, let δa be an element of Q, with $a \in S$. If for $i = 1, \dots, t$ the element δa is in $P_i^{(t+1-i)}$, then $\gamma_i(a) \geq 0$, and hence $a \in \mathcal{R}(I_t)$. Now observe that, for obvious reasons, Q does not contain P_i for $i = 1, \dots, t-1$. Also, Q contains P_t exactly if δ is the only t-minor of the matrix. But this is excluded by assumption. Therefore we may conclude that Q is a minimal prime of $(\delta)\mathcal{R}(I_t)$ and hence it is a height 1 prime of $\mathcal{R}(I_t)$. □

Theorem 6.2.9 *The divisor class group* $\mathrm{Cl}(\mathcal{R}(I_t))$ *of $\mathcal{R}(I_t)$ is free of rank t,*

$$\mathrm{Cl}(\mathcal{R}(I_t)) \cong \mathbb{Z}^t$$

with basis $\mathrm{cl}(P_1), \dots, \mathrm{cl}(P_t)$.

Proof. The conclusion follows from Proposition 6.2.6 and from a general result of Simis and Trung [59, Theorem 1.1]. Let us mention that one can derive the result also directly from Fossum [37, Theorem 7.1] and Proposition 6.2.8; to prove that $\mathrm{cl}(P_1), \ldots, \mathrm{cl}(P_t)$ are linear independent one may use equation (1). □

We have similar constructions and results for A_t. First define

$$\mathfrak{p} = \{F \in A_t : \gamma_2(F) \geq 1\}.$$

It is clear that $\mathfrak{p}$ is a prime ideal of A_t.

Lemma 6.2.10 *The ideal* $\mathfrak{p}$ *is prime of height* 1. *Moreover, one has* $\mathfrak{p}^{(j)} = \{F \in A_t : \gamma_2(F) \geq j\}$.

Proof. Let f be a $t+1$-minor of X. Set $g = f^t T^{t+1}$. By construction, $g \in V_t$ and $\gamma_2(g) = 1$ so that $g \in A_t$. Set

$$\mathfrak{q} = (f)S \cap A_t.$$

In other words, $\mathfrak{q} = \{fa \in A_t : a \in S\}$. Since f is a prime element in S, the ideal $\mathfrak{q}$ is prime. Furthermore we have $\mathfrak{p}\mathfrak{q}^t \subset (g) \subset \mathfrak{p} \cap \mathfrak{q}$. The second inclusion is trivial. As for the first, note that any generator of $\mathfrak{p}\mathfrak{q}^t$ can be written in the from gb, and then just evaluate γ_2 to show that b is in A_t. It follows that $\mathfrak{p}$ is a minimal prime of (g) (and hence a height 1 prime) unless it contains $\mathfrak{q}$. So we have to show that $\mathfrak{q} \not\subset \mathfrak{p}$. To this end let h be a minor of size $t-1$, then $hfT^2 \in \mathfrak{q}$ and $\gamma_2(hfT^2) = 0$.

The second statement is proved as in Lemma 6.2.5. □

As a corollary of the proof we get:

Corollary 6.2.11 *The ideal* $\mathfrak{q}$ *is a height* 1 *prime ideal of* A_t. *Furthermore* $\mathfrak{q}^{(j)} = (f^j)S \cap A_t$.

Proof. It suffices to check that $\mathfrak{q}$ does not contain $\mathfrak{p}$. Let f_1 be a $t+1$-minor of X different from f (it exists by our assumptions). Then $f_1^t T^{t+1}$ is in $\mathfrak{p}$, but not in $\mathfrak{q}$. The second statement is easy. □

Proposition 6.2.12 *Set* $A = A_t$. *Then one has*

$$V_t = \cap A_P$$

where the intersection is extended over all the height 1 *prime ideals* P *of* A_t *with* $P \neq \mathfrak{p}$.

Proof. We have to show that for every height 1 prime ideal P of V_t the intersection $P \cap A$ is a height 1 prime ideal of A. Let $g = f^t T^{t+1}$ be the element described in Lemma 6.2.10. By evaluating γ_2 one shows that $g^{t-1}\mu T$ is in A for every monomial μ of degree t in S. It follows that $A[g^{-1}] = V_t[g^{-1}]$. Hence it is enough to deal with the height 1 prime ideals of V_t containing g. Let $P = (f)S \cap V_t$. It is easy to check that $P^t \subset (g)V_t$ so that P is the only minimal prime of g in V_t. But $P \cap A = (f)S \cap A = \mathfrak{q}$, which has height 1 by Corollary 6.2.11. □

Theorem 6.2.13 *The divisor class group of A_t is free of rank* 1,

$$\mathrm{Cl}(A_t) \cong \mathbb{Z}$$

with basis $\mathrm{cl}(\mathfrak{q})$. *Furthermore we have* $\mathrm{cl}(\mathfrak{p}) = -t\mathrm{cl}(\mathfrak{q})$.

Proof. By Fossum [37, Theorem 7.1] and Proposition 6.2.12 we have an exact sequence:

$$0 \to \mathbb{Z}\mathrm{cl}(\mathfrak{p}) \to \mathrm{Cl}(A_t) \to \mathrm{Cl}(V_t) \to 0.$$

It is well-known that $\mathrm{Cl}(V_t)$ is isomorphic to $\mathbb{Z}/t\mathbb{Z}$ and that it is spanned by the class of the prime ideal P generated by the elements of the form $X_{11}\mu T$ where μ is a monomial in the X_{ij} of degree $t-1$. Note that $\mathrm{cl}(\mathfrak{p})$ is a torsion free element in $\mathrm{Cl}(A_t)$. This is because $\mathfrak{p}^{(j)} = \{F \in A_t : \gamma_2(F) \geq j\}$ cannot be principal; in fact, it contains all the elements of the form g^j where $g = f^t T^{t+1}$ and f is a $t+1$-minor. Now fix a $t+1$-minor f and set $P_1 = (f)S \cap V_t$, that is P_1 is the ideal generated by all the elements of the form $f\mu T^2$ where μ is a monomial of degree $t-1$. Evidently P_1 is isomorphic to P and hence $\mathrm{cl}(P_1)$ generates $\mathrm{Cl}(V_t)$. But $\mathfrak{q}$ is $P_1 \cap A_t$ and hence $\mathrm{cl}(\mathfrak{q})$ is the preimage of $\mathrm{cl}(P_1)$ with respect to the map $\mathrm{Cl}(A_t) \to \mathrm{Cl}(V_t)$. It follows that $\mathrm{cl}(\mathfrak{p})$ and $\mathrm{cl}(\mathfrak{q})$ generate $\mathrm{Cl}(A_t)$. By evaluating the function γ_2 one shows that $\mathfrak{q}^{(t)} \cap \mathfrak{p} = (g)A_t$. Hence $\mathrm{cl}(\mathfrak{p}) = -t\mathrm{cl}(\mathfrak{q})$. Consequently $\mathrm{Cl}(A_t)$ is generated by $\mathrm{cl}(\mathfrak{q})$, and $\mathrm{Cl}(A_t)$ is torsion free. □

6.3 Canonical modules of $\mathrm{in}_\tau(\mathcal{R}(I_t))$ and $\mathrm{in}_\tau(A_t)$

(**G**) The crucial step in determining the canonical modules of the rings $\mathcal{R}(I_t)$ and A_t is the description of the canonical modules of the semigroup rings $\mathrm{in}_\tau(\mathcal{R}(I_t))$ and $\mathrm{in}_\tau(A_t)$. We know that $\mathrm{in}_\tau(\mathcal{R}(I_t))$ and $\mathrm{in}_\tau(A_t)$ are normal (see Theorem 6.1.5 and Theorem 6.1.7) and hence their canonical modules are the vector spaces spanned by all monomials represented by integral points in the relative interiors of the corresponding cones.

To simplify notation we identify monomials of S with integral points of $\mathbb{R}^{mn+1}$. To a subset G of the lattice $\{1, \dots, m\} \times \{1, \dots, n\}$ we associate the ideal $P_G = (X_{ij} : (i,j) \notin G)$ of S and a linear form ℓ_G on $\mathbb{R}^{mn}$ defined by $\ell_G(X_{ij}) = 1$ if $(i,j) \notin G$ and $\ell_G(X_{ij}) = 0$ otherwise.

The initial ideal $\operatorname{in}_\tau(I_t(X))$ of $I_t(X)$ is the square free monomial ideal generated by the initial terms of the t-minors. Therefore it is the Stanley-Reisner ideal of the simplicial complex

$$\Delta_t = \{A \subseteq \{1,\dots,m\} \times \{1,\dots,n\} : A \text{ does not contain } t\text{-antichains}\}.$$

We know that Δ_t is a shellable simplicial complex; see Chapter 5. Denote by $\mathbf{F}_t$ the set of the facets of Δ_t. Then

$$\operatorname{in}_\tau(I_t(X)) = \bigcap_{F \in \mathbf{F}_t} P_F$$

where P_F denotes the ideal generated by the X_{ij} with $(i,j) \notin F$. The elements of $\mathbf{F}_t$ are described in Chapter 5 in terms of families of non-intersecting paths. We start by proving:

Proposition 6.3.1 *One has*

$$\operatorname{in}_\tau(I_t(X)^{(k)}) = \bigcap_{F \in \mathbf{F}_t} P_F^k. \tag{2}$$

By Theorem 4.3.4 we know that the initial ideal $\operatorname{in}_\tau(I_t(X)^{(k)})$ of $I_t(X)^{(k)}$ is generated by the monomials M with $\hat\gamma_t(M) \geq k$. Now a monomial $M = \prod_{i=1}^r X_{u_i v_i}$ is in P_F^k if and only if the cardinality of $\{i : (u_i, v_i) \notin F\}$ is $\geq k$. Equivalently, M is in P_F^k if and only if the cardinality of $\{i : (u_i, v_i) \in F\}$ is $\leq \deg(M) - k$. If we set

$$w_t(M) = \max\{|A| : A \subseteq \{1,\dots,s\} \text{ and } \{(u_i,v_i) : i \in A\} \in \Delta_t\}$$

then we have that a monomial M is in $\bigcap_{F\in\mathbf{F}_t} P_F^k$ if and only if $w_t(M) \leq \deg(M) - k$, that is, $\deg(M) - w_t(M) \geq k$. Now Proposition 6.3.1 follows from:

Lemma 6.3.2 *Let M be a monomial. Then* $\hat\gamma_t(M) + w_t(M) = \deg(M)$.

We reduce this lemma to a combinatorial statement on sequences of integers. Given such a sequence v we define $w_t(v)$ to be the cardinality of the longest subsequence of v which does not contain an increasing subsequence of length t, that is,

$$w_t(v) = \max\{\text{length}\,(c) : c \text{ is a subsequence of } v \text{ and } \gamma_t(c) = 0\}.$$

Let $M = \prod_{i=1}^r X_{u_i v_i}$ be a monomial. We may order the indices such that $u_i \leq u_{i+1}$ for every i and $v_{i+1} \geq v_i$ whenever $u_i = u_{i+1}$. (We have already considered this rearrangement in Chapter 3.) Then the t-diagonals dividing M correspond to increasing subsequences of length t of the sequence v, and $w_t(M) = w_t(v)$. Since $w_t(M)$ depends only on the sequence v, identically as in Chapter 3, we may assume that $u_i = i$ for every i. Then, by exchanging the role between the u_i's and the v_i's we may also assume that the v_i are distinct integers. Summing up, it suffices to show the following:

Lemma 6.3.3 *One has* $\hat{\gamma}_t(v) + w_t(v) = \text{length}(v)$ *for every sequence* v *of distinct integers.*

Proof. Let $P = \text{INSERT}(v)$ be the tableau obtained from v by the insertion algorithm. We have that $w_t(v)$ is equal to the maximal length of a subsequence of v which can be decomposed into $t-1$ decreasing subsequences. Therefore $w_t(v) = d_{t-1}(v)$; see the notation before Green's theorem in Chapter 3. On the other hand, by Theorem 4.3.3 we know that $\hat{\gamma}_t(v)$ is equal to $\gamma_t(P)$ which is the sum of the length of the columns of P of index $\geq t$. Therefore $\hat{\gamma}_t(v)+w_t(v)$ is equal to the number of entries of P which is the length of v. □

We know that $\text{in}_\tau(I_t(X)^k) = \bigcap_{j=1}^t \text{in}_\tau(I_j(X)^{(k(t+1-j))})$ (in non-exceptional characteristic) and hence, taking into consideration Proposition 6.3.1, we have:

$$\text{in}_\tau(I_t(X)^k) = \bigcap_{j=1}^{t} \bigcap_{F\in\mathbf{F}_j} P_F^{k(t+1-j)}. \tag{3}$$

Since the powers of the ideal P_F are P_F-primary we have that (3) is indeed a primary decomposition of $\text{in}_\tau(I_t(X)^k)$.

For every $F \in \mathbf{F}_i$ we extend the linear form ℓ_F to the linear form L_F on $\mathbb{R}^{mn+1}$ by setting $L_F(T) = -(t+1-i)$. Then the equations (2) and (3) imply:

Lemma 6.3.4 *A monomial* N *belongs to* $\text{in}_\tau(\mathcal{R}(I_t))$ *if and only if it has non-negative exponents and* $L_F(N) \geq 0$ *for every* $F \in \mathbf{F}_i$ *and* $i = 1,\dots,t$.

Theorem 6.3.4 can be interpreted as a description of the normal semigroup of monomials in the initial algebra of the Rees algebra of $I_t(X)$ in terms of linear homogeneous inequalities. It follows from the general theory (see Bruns and Herzog [19, Chapter 6]) that the canonical module $\omega(\text{in}_\tau(\mathcal{R}(I_t)))$ of $\text{in}_\tau(\mathcal{R}(I_t))$ is the semigroup ideal of $\mathcal{R}(I_t)$ generated by the monomials N with all exponents ≥ 1 and $L_F(N) \geq 1$ for every $F \in \mathbf{F}_i$ and $i = 1,\dots,t$.

Let $\mathcal{X}$ denote the product of all the indeterminates X_{ij} with $(i,j) \in \{1,\dots,m\}\times\{1,\dots,n\}$. The canonical module $\omega(\text{in}_\tau(\mathcal{R}(I_t)))$ has a description in terms of $\mathcal{X}$ and the functions $\hat{\gamma}_i$:

Lemma 6.3.5 *The canonical module* $\omega(\text{in}_\tau(\mathcal{R}(I_t)))$ *of* $\text{in}_\tau(\mathcal{R}(I_t))$ *is the ideal*

$$\{F \in S : \mathcal{X}T \mid F \text{ in } S \text{ and } \hat{\gamma}_i(F) \geq 1 \text{ for every } i = 1, \dots, t\}$$

of $\text{in}_\tau(\mathcal{R}(I_t))$.

Proof. Let $N = MT^k$ be a monomial, where M is a monomial in X_{ij}. Then, for a given i, $L_F(N) \geq 1$ for every $F \in \mathbf{F}_i$ if and only if $\ell_F(M) \geq k(t+1-i)+1$ for every $F \in \mathbf{F}_i$. By (2) this is equivalent to $M \in \text{in}_\tau(I_i^{(k(t+1-i)+1)})$ which in turn is equivalent to $\hat{\gamma}_i(M) \geq k(t+1-i)+1$. Summing up, $L_F(N) \geq 1$ for every $F \in \mathbf{F}_i$ and $i = 1, \dots, t$ if and only if $\hat{\gamma}_i(N) \geq 1$ for every $i = 1, \dots, t$. □

Similarly the canonical module $\omega(\text{in}_\tau(A_t))$ has a description in terms of the function $\hat{\gamma}_2$:

Lemma 6.3.6 *The canonical module* $\omega(\text{in}_\tau(A_t))$ *of* $\text{in}_\tau(A_t)$ *is the ideal*

$$\{F \in V_t : \mathcal{X}T \mid F \text{ in } S \text{ and } \hat{\gamma}_2(F) \geq 1\}$$

of $\text{in}_\tau(A_t)$.

(**A**) We already know that $\text{in}_\tau(\mathcal{R}(I_t))$ and $\text{in}_\tau(A_t)$ are normal (see Theorem 6.1.12 and Theorem 6.1.14) and hence its canonical module is the vector space spanned by all monomials represented by integral points in the relative interior of the corresponding cone; see [19, Ch. 6].

In the following we identify X_{ij} with the point (i,j) on the plane. On the set $X = \{(i,j) : 1 \leq i < j \leq n\}$ we introduce the partial order $(i,j) \leq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$. The initial ideal $\text{in}_\tau(I_t(X))$ of $I_t(X)$ is the Stanley-Reisner ideal of the simplicial complex

$$\Delta_t = \{A \subseteq X : A \text{ does not contain } t\text{-antichains}\};$$

see Chapter 5. Denote by $\mathbf{F}_t$ the set of the facets of Δ_t. Then

$$\text{in}_\tau(I_t(X)) = \bigcap_{F \in \mathbf{F}_t} P_F$$

where P_F denotes the ideal generated by the X_{ij} with $(i,j) \notin F$.

Proposition 6.3.7 *One has*

$$\text{in}_\tau(I_t^{(k)}) = \bigcap_{F \in \mathbf{F}_t} P_F^k. \tag{4}$$

Proof. As in part (G), the proposition reduces to a combinatorial statement on sequences of integers. Given such a sequence b we define $w_t(b)$ to be the length of the longest subsequence of b which does not contain a decreasing subsequence of length t, that is,

$$w_t(b) = \max\{\text{length}\,(c) : c \text{ is a subsequence of } b \text{ and } \gamma_t(c) = 0\}.$$

Actually, we have to show that

$$\hat{\gamma}_t(b) + w_t(b) = \text{length}\,(b)$$

for every sequence b of distinct integers.

Let $P = \text{INSERT}(b)$. By virtue of Greene's theorem 3.2.2, the sum $d_k(b)$ of the lengths of the first k rows of P is the length of the longest subsequence of b that can be decomposed into k decreasing subsequences. Therefore $w_t(b) = d_{t-1}(b)$. On the other hand, by Theorem 4.3.10 we know that $\hat{\gamma}_t(b)$ is equal to $\gamma_t(P)$ which is the sum of the length of the rows of P of index $\geq t$. Therefore $\hat{\gamma}_t(b) + w_t(b)$ is equal to the number of entries of P which is the length of b. □

Proposition 6.3.7 has the following consequence:

Corollary 6.3.8 *One has*

$$\text{in}_\tau(I_t(X)^k) = \bigcap_{j=1}^{t} \bigcap_{F \in \mathbf{F}_j} P_F^{k(t+1-j)}. \tag{5}$$

Now small changes of the arguments in part (G) yield very similar results for pfaffians. First let us identify the monomials of S with integral points of $\mathbb{R}^{n(n-1)/2+1}$. To a subset G of the lattice X we associate the ideal $P_G = (X_{ij} : (i,j) \notin G)$ of S and a linear form ℓ_G on $\mathbb{R}^{n(n-1)/2}$ defined by $\ell_G(X_{ij}) = 1$ if $(i,j) \notin G$ and $\ell_G(X_{ij}) = 0$ otherwise.

For every $F \in \mathbf{F}_i$ we extend the linear form ℓ_F to the linear form L_F on $\mathbb{R}^{n(n-1)/2+1}$ by setting $L_F(T) = -(t+1-i)$. Then the equations (4) and (5) imply

Lemma 6.3.9 *A monomial N belongs to* $\text{in}_\tau(\mathcal{R}(I_t))$ *if and only if it has non-negative exponents and* $L_F(N) \geq 0$ *for every* $F \in \mathbf{F}_i$ *and* $i = 1, \ldots, t$.

Lemma 6.3.9 provides the description of the semigroup in terms of linear homogeneous inequalities. Let $\mathcal{X}$ denote the product of all the indeterminates X_{ij} with $(i,j) \in X$. Then the canonical module $\omega(\text{in}_\tau(\mathcal{R}(I_t)))$ has the following description in terms of $\mathcal{X}$ and the functions $\hat{\gamma}_i$:

Lemma 6.3.10 *The canonical module* $\omega(\mathrm{in}_\tau(\mathcal{R}(I_t)))$ *of* $\mathrm{in}(\mathcal{R}(I_t))$ *is the ideal*

$$\{F \in S : \mathcal{X}T \mid F \text{ in } S \text{ and } \hat{\gamma}_i(F) \geq 1 \text{ for every } i = 1, \dots, t\}$$

of $\mathrm{in}_\tau(\mathcal{R}(I_t))$.

Similarly we get:

Lemma 6.3.11 *The canonical module* $\omega(\mathrm{in}_\tau(A_t))$ *of* $\mathrm{in}_\tau(A_t)$ *is the ideal*

$$\{F \in V_t : \mathcal{X}T \mid F \text{ in } S \text{ and } \hat{\gamma}_2(F) \geq 1\}$$

of $\mathrm{in}_\tau(A_t)$.

6.4 The canonical classes of $\mathcal{R}(I_t)$ and A_t

(**G**) In this section we will describe the canonical modules of $\mathcal{R}(I_t)$ and of A_t and determine the canonical classes. In order to do this we have to proceed to a "de-initialization" of the results in Section 6.3. To this end we need the deformation lemma of Bruns and Conca [16, Lemma 4.1].

Lemma 6.4.1 *Let* $R = K[X_1, \dots, X_n]$ *be a polynomial ring equipped with a term order and with a grading induced by positive weights* $\deg(X_i) = v_i$. *Let* B *be a finitely generated* K*-subalgebra of* R *generated by homogeneous polynomials and* J *be a homogeneous ideal of* B. *Denote by* $\mathrm{in}(B)$ *and* $\mathrm{in}(J)$ *the initial algebra and the initial ideal of* B *and* J *respectively. Then we have:*
(a) *If* $\mathrm{in}(B)$ *is finitely generated and* $\mathrm{in}(B)/\mathrm{in}(J)$ *is Cohen-Macaulay, then* B/J *is Cohen-Macaulay.*
(b) *If* $\mathrm{in}(B)$ *is finitely generated and Cohen-Macaulay and* $\mathrm{in}(J)$ *is the canonical module of* $\mathrm{in}(B)$ *(up to shift) then* B *is Cohen-Macaulay and* J *is the canonical module of* B *(up to the same shift).*

Proof. (a) Let $f_1, \dots, f_k$ be a Sagbi basis of B and $g_1, \dots, g_h$ a Gröbner basis of J. We may assume that these polynomials are monic and homogeneous. Consider the presentation $K[Y_1, \dots, Y_k]/I \cong B$ of B obtained by mapping Y_i to f_i and the presentation $K[Y_1, \dots, Y_k]/I_1 \cong \mathrm{in}(B)$ of $\mathrm{in}(B)$ obtained by mapping Y_i to $\mathrm{in}(f_i)$. Let $h_1, \dots, h_p$ a system of binomial generators for the toric ideal I_1, say $h_i = Y^{a_i} - Y^{b_i}$. Then for each i we have expressions $f^{a_i} - f^{b_i} = \sum \lambda_{ij} f^{c_{ij}}$ with $\lambda_{ij} \in K \setminus \{0\}$ and $\mathrm{in}(f^{c_{ij}}) < \mathrm{in}(f^{a_i})$ for every i, j. It is known that the polynomials

$$Y^{a_i} - Y^{b_i} - \sum \lambda_{ij} Y^{c_{ij}} \tag{4}$$

generate I. For each g_i we may take a presentation $g_i = f^{d_i} + \sum \delta_{ij} f^{d_{ij}}$ with $\delta_{ij} \in K \setminus \{0\}$ and $\mathrm{in}(f^{d_{ij}}) < \mathrm{in}(f^{d_i})$ for all i, j. Then, by construction, the preimage of J in $K[Y]/I$ is generated by the elements

$$Y^{d_i} + \sum \delta_{ij} Y^{d_{ij}}, \tag{5}$$

and the preimage of $\mathrm{in}(J)$ in $K[Y]/I_1$ is generated by the Y^{d_i}. Hence B/J is the quotient of $K[Y]$ defined by the ideal H that is generated by the polynomials (4) and (5), and $\mathrm{in}(B)/\mathrm{in}(J)$ is the quotient of $K[Y]$ defined by the ideal H_1 that is generated by the polynomials h_i and Y^{d_i}.

If we can find a positive weight w on $K[Y]$ such that $\mathrm{in}_w(H) = H_1$, then there is 1-parameter flat family with special fiber $\mathrm{in}(B)/\mathrm{in}(J)$ and general fiber B/J. This implies that B/J is Cohen-Macaulay, provided $\mathrm{in}(B)/\mathrm{in}(J)$ is. Let us define w. First consider a positive weight α on $K[X]$ such that $\mathrm{in}_\alpha(f^{c_{ij}}) < \mathrm{in}_\alpha(f^{a_i})$ for every i, j and $\mathrm{in}_\alpha(f^{d_{ij}}) < \mathrm{in}_\alpha(f^{d_i})$ for every i and j. That such an α exists is a well-known property of monomial orders; for instance, see Sturmfels [61, Proof of Cor. 1.11]. Then we define w as the "preimage" of α in the sense that we put $w(Y_i) = \alpha(\mathrm{in}(f_i))$. It is clear, by construction, that the initial forms of the polynomials (4) and (5) with respect to w are exactly the h_i and the Y^{d_i}.

This proves that $\mathrm{in}_w(H)$ contains H_1. But H_1 and H have the same Hilbert function by construction, and H and $\mathrm{in}_w(H)$ have the same Hilbert function because they have the same initial ideal if we refine w to a term order; for instance see Sturmfels [61, Prop. 1.8]. Here we consider Hilbert functions with respect to the original graded structure induced by the weights v_i. It follows that $\mathrm{in}_w(H) = H_1$ and we are done.

(b) That B is Cohen-Macaulay follows from Conca, Herzog, and Valla [27]. Since B is a Cohen-Macaulay positively graded K-algebra which is a domain, to prove that J is the canonical module of B it suffices to show that J is a maximal Cohen-Macaulay module whose Hilbert series satisfies the relation $H_J(t) = (-1)^d t^k H_B(t^{-1})$ for some integer k where $d = \dim B$; see Bruns and Herzog [19, Theorem 4.4.5 and Corollary 4.4.6].

The relation $H_J(t) = (-1)^d t^k H_B(t^{-1})$ holds since by assumption the corresponding relation holds for the initial objects and Hilbert series do not change by taking initial terms. So it is enough to show that J is a maximal Cohen-Macaulay module. But $\mathrm{in}(J)$ is a height 1 ideal since it is the canonical module (see Bruns and Herzog [19, Prop. 3.3.18]), and hence also J has height 1. Therefore it suffices to show that B/J is a Cohen-Macaulay ring. But this follows from (a) since $\mathrm{in}(B)/\mathrm{in}(J)$ is Cohen-Macaulay (it is even Gorenstein) [19, Prop. 3.3.18]. □

We also need the following lemma whose part (b) asserts that $\mathcal{X}$ is a "linear" element for the functions $\hat{\gamma}_i$.

Lemma 6.4.2 (a) *One has* $\hat{\gamma}_i(\mathcal{X}) = (m-i+1)(n-i+1)$.
(b) *Let* M *be any monomial in the indeterminates* X_{ij}. *Then* $\hat{\gamma}_i(\mathcal{X}M) = \hat{\gamma}_i(\mathcal{X}) + \hat{\gamma}_i(M)$ *for every* $i = 1, \dots, \min(m, n)$.

Proof. Let M be a monomial in the X_{ij}'s. We know that $\hat{\gamma}_i(M) \geq k$ if and only if $M \in \operatorname{in}_\tau(I_i(X)^{(k)})$. From equation (2) we deduce that

$$\hat{\gamma}_i(M) = \inf\{\ell_F(M) : F \in \mathbf{F}_i\}.$$

Note that Δ_i is a pure simplicial complex of dimension equal to the dimension of $R_i(X)$, the classical determinantal ring defined by $I_i(X)$, minus 1. It follows that $\ell_F(\mathcal{X}) = (m-i+1)(n-i+1)$ for every facet F of Δ_i. In particular, $\hat{\gamma}_i(\mathcal{X}) = (m-i+1)(n-i+1)$.

Since $\ell_F(NM) \geq \ell_F(N) + \ell_F(M)$ for all monomials N, M and for every F, we have $\hat{\gamma}_i(MN) \geq \hat{\gamma}_i(N) + \hat{\gamma}_i(M)$. Conversely, let G be a facet of Δ_i such that $\hat{\gamma}_i(M) = \ell_G(M)$. Then $\ell_G(\mathcal{X}M) = \ell_G(\mathcal{X}) + \ell_G(M) = \hat{\gamma}_i(\mathcal{X}) + \hat{\gamma}_i(M)$. Hence $\hat{\gamma}_i(MN) \leq \hat{\gamma}_i(N) + \hat{\gamma}_i(M)$, too. □

Now assume for simplicity that $m \leq n$. Let us consider a product of minors μ such that $\operatorname{in}_\tau(\mu) = \mathcal{X}$ and $\gamma_i(\mu) = \hat{\gamma}_i(\mathcal{X})$. Since we have already computed $\hat{\gamma}_i(\mathcal{X})$ (see Lemma 6.4.2), we can determine the shape of μ, which turns out to be $1^2, 2^2, \dots, (m-1)^2, m^{(n-m+1)}$. In other words, μ must be the product of 2 minors of size 1, 2 minors of size 2, ..., 2 minors of size $m-1$ and $n-m+1$ minors of size m. It is then not difficult to show that μ is uniquely determined, the 1-minors are $[m|1]$ and $[1|n]$, the 2-minors are $[m-1, m|1, 2]$ and $[1, 2|n-1, n]$ and so on.

We have:

Theorem 6.4.3 (a) *The canonical module of* $\mathcal{R}(I_t)$ *is the ideal*

$$J = \{F \in S : \mu T \mid F \textit{ in } S \textit{ and } \gamma_i(F) \geq 1 \textit{ for } i = 1, \dots, t\}.$$

(b) *The canonical module of* A_t *is the ideal*

$$J_1 = \{F \in V_t : \mu T \mid F \textit{ in } S \textit{ and } \gamma_2(F) \geq 1\}.$$

Proof. By virtue of Lemma 6.4.1 it suffices to show that $\operatorname{in}_\tau(J)$ and $\operatorname{in}_\tau(J_1)$ are the canonical modules of $\operatorname{in}_\tau(\mathcal{R}(I_t))$ and $\operatorname{in}_\tau(A_t)$ respectively. A description of the canonical modules of $\operatorname{in}_\tau(\mathcal{R}(I_t))$ and $\operatorname{in}_\tau(A_t)$ has been given in Section 6.3. Therefore it is enough to check that $\operatorname{in}_\tau(J)$ is exactly the ideal described in 6.3.5 and $\operatorname{in}_\tau(J_1)$ is the ideal described in 6.3.6. Note that we may write

$$J = \mu T\{F \in S : \gamma_i(F) \geq 1 - \gamma_i(\mu T) \text{ for } i = 1, \dots, t\}$$

and

$$J_1 = \mu T\{F \in S : \gamma_2(F) \geq 1 - \gamma_2(\mu T)\} \cap V_t.$$

Furthermore, by virtue of Lemma 6.4.2,

$$\omega(\operatorname{in}_\tau(\mathcal{R}(I_t))) = \mathcal{X}T\{F \in S : \hat{\gamma}_i(F) \geq 1 - \hat{\gamma}_i(\mathcal{X}T) \text{ for } i = 1, \dots, t\}$$

and

$$\omega(\operatorname{in}_\tau(A_t)) = \mathcal{X}T\{F \in S : \hat{\gamma}_2(F) \geq 1 - \hat{\gamma}_2(\mathcal{X}T)\} \cap V_t.$$

Since, by the very definition of μ, we have $\operatorname{in}_\tau(\mu) = \mathcal{X}$ and $\gamma_i(\mu) = \hat{\gamma}_i(\mathcal{X})$, it suffices to show that

$$\operatorname{in}_\tau(\{F \in S : \gamma_i(F) \geq j\}) = \{F \in S : \hat{\gamma}_i(F) \geq j\}.$$

But this has (essentially) been proved in Chapter 4. □

Now we determine the canonical class of $\mathcal{R}(I_t)$.

Theorem 6.4.4 *The canonical class of $\mathcal{R}(I_t)$ is given by*

$$\operatorname{cl}(\omega(\mathcal{R}(I_t))) = \sum_{i=1}^t \Big(2 - (m-i+1)(n-i+1) + t - i\Big)\operatorname{cl}(P_i) = \operatorname{cl}(I_t\mathcal{R}(I_t)) + \sum_{i=1}^t (1 - \operatorname{height} I_i(X))\operatorname{cl}(P_i)$$

Proof. Note that the second formula for $\omega(\mathcal{R}(I_t))$ follows from the first, since $\operatorname{height} I_i(X) = (m-i+1)(n-i+1)$ and $\operatorname{cl}(I_t(X)\mathcal{R}(I_t)) = \sum_{i=1}^t (t-i+1)\operatorname{cl}(P_i)$ by Proposition 6.2.6.

We have seen that

$$\omega(\mathcal{R}(I_t)) = \mu T\{F \in S : \gamma_i(F) \geq 1 - \gamma_i(\mu T) \text{ for } i = 1, \dots, t\}.$$

We can get rid of μT and obtain a representation of $\omega(\mathcal{R}(I_t))$ as a fractional ideal, namely,

$$\omega(\mathcal{R}(I_t)) = \{F \in S : \gamma_i(F) \geq 1 - \gamma_i(\mu T) \text{ for } i = 1, \dots, t\}.$$

It follows that $\omega(\mathcal{R}(I_t)) = \cap P_i^{(1-\gamma_i(\mu T))}$. As $\gamma_i(\mu T) = (m-i+1)(n-i+1) - (t+1-i)$, we are done. □

For A_t the situation is slightly more difficult since $\mu \notin A_t$ in general. Therefore we need an auxiliary lemma:

Lemma 6.4.5 *Let δ_j be a j-minor of X and $\mathfrak{q}_j = (\delta_j)S \cap A_t$. Then $\mathfrak{q}_j$ is a height 1 prime ideal of A_t and $\operatorname{cl}(\mathfrak{q}_j) = (j-t)\operatorname{cl}(\mathfrak{q})$.*

Proof. Note that $\mathfrak{q}_{t+1} = \mathfrak{q}$ by definition. Let ν be a product of minors. We say that ν has *tight shape* if its degree is divisible by t and it has exactly $\deg(\nu)/t$ factors. In other words, ν has tight shape if $\gamma_1(\nu T^k) = 0$ and $\gamma_2(\nu T^k) = 0$

where $k = \deg(\nu)/t$. Let ν be a product of minors with tight shape and $k = \deg(\nu)/t$. Note that

$$(\nu)S \cap A_t = (\nu T^k)A_t.$$

Now fix $j \le t$ and set $k = t-j+1$; the product $\nu = \delta_j\delta_{t+1}^{t-j}$ has tight shape and hence $\mathfrak{q}_j \cap \mathfrak{q}_{t+1}^{(t-j)} = (\nu T^k)A_t$. Note that, for obvious reasons, $\mathfrak{q}_j$ does not contain $\mathfrak{q}$. It follows that $\mathfrak{q}_j$ is a prime ideal of height 1 and that $\mathrm{cl}(\mathfrak{q}_j) = (j-t)\mathrm{cl}(\mathfrak{q})$. Now take $j > t$ and set $k = j - t + 1$; the product $\nu = \delta_j\delta_{t-1}^{j-t}$ has tight shape and hence, as above, we conclude that $\mathfrak{q}_j$ is prime of height 1 and $\mathrm{cl}(\mathfrak{q}_j) = (t-j)\mathrm{cl}(q_{t-1})$. Since we know already that $\mathrm{cl}(q_{t-1}) = -\mathrm{cl}(\mathfrak{q})$, we are done. □

Now we can prove

Theorem 6.4.6 *Then the canonical class of A_t is given by*

$$\mathrm{cl}(\omega(A_t)) = (mn - tm - tn)\mathrm{cl}(\mathfrak{q}).$$

Proof. Assume that $m \le n$. Set $W = (\mu T)S \cap A_t$. We have seen that

$$\omega(A_t) = W \cap \mathfrak{p}.$$

Consequently

$$\mathrm{cl}(\omega(A_t)) = \mathrm{cl}(W) + \mathrm{cl}(\mathfrak{p}) = \mathrm{cl}(W) - t\mathrm{cl}(\mathfrak{q}).$$

Note that W can be written as the intersection of $(\mu)S \cap A_t$ and $(T)S \cap A_t$. But the latter is the irrelevant maximal ideal of A_t, whence $W = (\mu)S \cap A_t$. Further $(\mu)S \cap A_t$ can be written as an intersection of ideals $\mathfrak{q}_j$. Taking into consideration the shape of μ and Lemma 6.4.5, we have

$$\mathrm{cl}(W) = \Big(\sum_{j=1}^{m-1} 2(j-t) + (n-m+1)(m-t)\Big)\mathrm{cl}(\mathfrak{q}).$$

Summing up, we get the desired result. □

As a corollary we have

Theorem 6.4.7 *The ring A_t is Gorenstein if and only if $mn = t(m+n)$.*

Note that we assume $1 < t < \min(m,n)$ and $m \ne n$ if $t = \min(m,n) - 1$ in the theorem. We have observed in the introduction that A_t is indeed Gorenstein (and even factorial) in these remaining cases.

Remark 6.4.8 It is also possible to derive Theorem 6.4.6 from Theorem 6.4.4 by a suitable generalization of [21, Lemma (8.10)].

(**A**) Let us consider a product of pfaffians π such that $\text{in}_\tau(\pi) = \mathcal{X}$ and $\gamma_i(\pi) = \hat{\gamma}_i(\mathcal{X})$.

Lemma 6.4.9 (a) $\hat{\gamma}_i(\mathcal{X}) = (n-2i+1)(n-2i+2)/2$.
(b) *Let M be any monomial in the X_{ij}'s. Then* $\hat{\gamma}_i(\mathcal{X}M) = \hat{\gamma}_i(\mathcal{X}) +$ *$hat\gamma_i(M)$ for every $i = 1, \ldots, [n/2]$.*

Since we know $\hat{\gamma}_i(\mathcal{X})$, we can determine the shape of π. It turns out to be $1^4, 2^4, \ldots, ((n-2)/2)^4, (n/2)^1$ for n even, and $1^4, 2^4, \ldots, ((n-3)/2)^4, ((n-1)/2)^3$ for n odd. In other words, π must be the product of 4 pfaffians of size 2, 4 pfaffians of size 4, and so on. It is not difficult to see that π is uniquely determined. Similar to part (G) we can prove

Theorem 6.4.10 (a) *The canonical module of $\mathcal{R}(I_t)$ is the ideal*

$$J = \{F \in S : \pi T \mid F \text{ in } S \text{ and } \gamma_i(F) \geq 1 \text{ for } i = 1, \ldots, t\}.$$

(b) *The canonical module of A_t is the ideal*

$$J_1 = \{F \in V_t : \pi T \mid F \text{ in } S \text{ and } \gamma_2(F) \geq 1\}.$$

Now we determine the canonical class of $\mathcal{R}(I_t)$.

Theorem 6.4.11 *The canonical class of $\mathcal{R}(I_t)$ is given by*

$$\operatorname{cl}(\omega(\mathcal{R}(I_t))) = \sum_{i=1}^{t}\Big(2 - (n-2i+1)(n-2i+2)/2 + t - i\Big)\operatorname{cl}(P_i) = \operatorname{cl}(I_t\mathcal{R}(I_t)) + \sum_{i=1}^{t}(1 - \operatorname{height} I_i(X))\operatorname{cl}(P_i)$$

Set $\mathfrak{q} = (f)S \cap A_t$, where f is an $2t+2$-pfaffian of X.

Theorem 6.4.12 *Then the canonical class of A_t is given by*

$$\operatorname{cl}(\omega(A_t)) = \left(\binom{n}{2} - 2t(n-1)\right)\operatorname{cl}(\mathfrak{q}).$$

As a corollary we have:

Theorem 6.4.13 *The ring A_t is Gorenstein if and only if $n = 4t$.*

Recall that we assume $2 < 2t < n-2$ in the theorem. We have noted in the introduction that A_t is Gorenstein (and even factorial) in these remaining cases.

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